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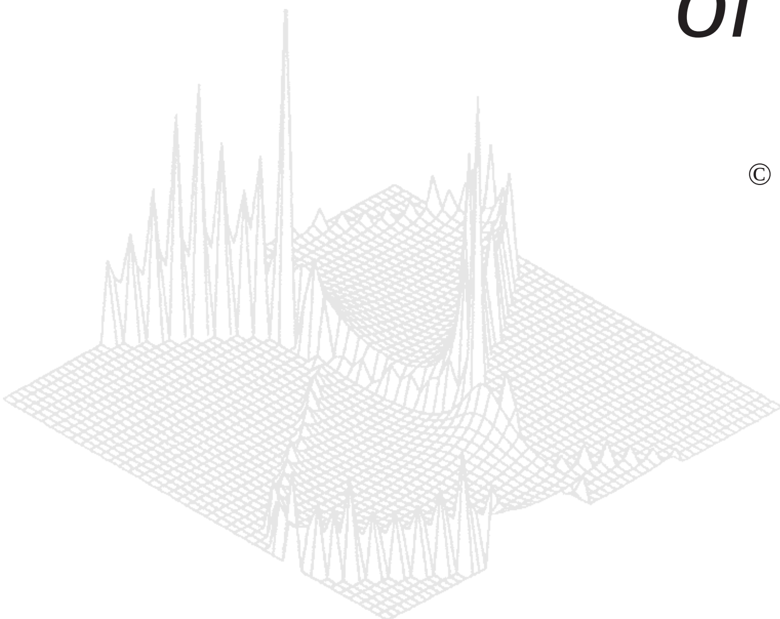

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Two Remarks Concerning the Signs of the Light Quark Masses

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Abstract

Partial conservation of the *nine* axial currents A_μ^β for $\beta = 0, 1, ..., 8$ applied to the vacuum expectation values of the equal-time commutators of the nine axial charges $Q_A^\alpha(x_0)$ for $\alpha = 0, 1, ..., 8$ and the corresponding axial current divergences $\partial^\mu A_\mu^\beta$ implies that the non-vanishing current quark masses m_u, m_d, m_s of the light quarks u, d, s have the same sign. Under the more realistic assumption of partial conservation of only the eight axial currents of $SU(3) \otimes SU(3)$, i.e. $\alpha = 1, ..., 8$ and $\beta = 1, ..., 8$ in the above, inequalities for the quark masses follow. They are trivially fulfilled if the three light quark masses have the same sign and imply, for non-vanishing quark masses m_u, m_d, m_s, that at least two of these masses have the same sign as their sum $m_u + m_d + m_s$. If one of the three quark masses vanishes, one of the other two might also. If both do not, they have the same sign (the same, of course, as their sum). Our assumptions include the standard *vector* SU(3) symmetry of the vacuum.

1. Introduction and Conclusions

Since quarks are confined, common sense arguments that all masses must have the same (positive) sign cannot be applied to them. The present paper derives restrictions on the signs of the light current quark masses from the principles of quantum field theory, the standard model and Goldstone chiral symmetry breaking.

Let us more precisely and maybe unrealistically first assume that in computing the vacuum expectation values (VEV) $\langle [Q_A^\alpha(x_0), \partial^\mu A_\mu^\beta(x)] \rangle_0$ of the σ commutators of the *nine* axial charges Q_A^α with the *nine* axial current divergencies $\partial^\mu A_\mu^\beta$, we may use partial conservation of these currents. Thus, for the present purpose the divergencies of the 16 vector and axial vector currents of $SU(3) \otimes SU(3)$ *and* of the $SU(3)$-neutral ninth axial current A_μ^0 may be computed from the equation of motion of the standard model, or equivalently, from equation (8) together with the definition of $u(x)$ in equation (3). The eight *vector* charges $Q_V^a(x_0)$ for $a = 1, ..., 8$ furthermore (almost) annihilate the vacuum such that the VEVs of the vector σ-terms may be neglected compared to the VEVs of the axial σ-terms.

This feature of chiral symmetry breaking is absolutely essential for our conclusions. For example, the replacement of the field ψ_u of the u-quark by $\gamma_5\psi_u$ flips the sign of m_u and otherwise leaves the QCD Lagrangian invariant. This is however not a counterexample to our claims since under this transformation some

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of the vector currents such as $\bar{\psi}_u\gamma_\mu\psi_d$ are taken into axial currents ($\bar{\psi}_u\gamma_\mu\gamma_5\psi_d$ in our example) such that, in the new basis, the standard approximate $SU(3)$ symmetry of the vacuum, which we assume, is replaced by a symmetry that is generated by a mixture of vector and axial charges.*

From the vacuum structure of the standard model and the fact that $(-i)$-times the VEV of a σ-term of a charge $Q(x_0) = \int d^3x J_0(x)$ and the divergence $\partial^\mu J_\mu$ of the corresponding hermitean current J_μ is non-negative,

$$-i\langle[Q, \partial^\mu J_\mu]\rangle_0 \geq 0\,, \tag{1}$$

sum rules are derived that are fulfilled if and only if the non-vanishing light quark current quark masses m_u, m_d and m_s have the same (positive or negative) sign.

Restricting the above assumptions more realistically to the 16 vector and axial vector currents of $SU(3) \otimes SU(3)$ and assuming that the sum $m_u + m_d + m_s$ of the three quark masses is positive, we obtain the relations

$$m_u + m_d \geq 0\,, \tag{2a}$$

$$m_u + m_s \geq 0\,, \tag{2b}$$

$$m_d + m_s \geq 0\,, \tag{2c}$$

$$m_u + m_d + 4m_s \geq 0\,, \tag{2d}$$

$$m_u m_d + m_u m_s + m_d m_s \geq 0\,, \tag{2e}$$

for the quark masses. They are trivially fulfilled if the masses of all three quarks are non-negative and imply, for example, that at least two of the three masses must be just that (i.e. non-negative). If one of the three masses vanishes, a second one might. If not, the non-vanishing masses are both positive. In the presumably academic case that the sum of the three masses is negative, the greater-or-equal signs in equations (2a)–(2d) are inverted, whereas (2e) remains unaltered. The conclusions remain very much the same except that 'non-negative' is replaced by 'non-positive' and 'positive' by 'negative'. The case of a vanishing sum of the quark masses is not discussed here.

2. Reminders

It is the purpose of the present section to remind the reader of the rather old-fashioned methods, going back to Gell-Mann *et al.* (1968), of chiral symmetry breaking that we use. The light quark mass term (averaging over colour is always understood)

$$u(x) = m_u\bar{\psi}_u(x)\psi_u(x) + m_d\bar{\psi}_d(x)\psi_d(x) + m_s\bar{\psi}_s(x)\psi_s(x) \tag{3}$$

of the Hamiltonian density of the standard model *and* chiral symmetry breaking (see Leutwyler 1994 for a discussion) is written in the form

* A much more detailed discussion of the relation of the quark masses to the Goldstone nature of the vacuum, which can be traced back to Dashen (1971), is contained in Leutwyler (1983). An early review is Pagels (1975). I thank G. Ecker for asking a relevant question, an anonymous referee for hints, and H. Leutwyler for sending me a copy of Leutwyler (1983).

$$u(x) = c_0 u_0(x) + c_3 u_3(x) + c_8 u_8(x)\,, \tag{4}$$

where the definitions

$$c_0 = (m_u + m_d + m_s)/\sqrt{6}\,, \tag{5a}$$

$$c_3 = (m_u - m_d)/2\,, \tag{5b}$$

$$c_8 = (m_u + m_d - 2m_s)/(2\sqrt{3})\,, \tag{5c}$$

together with

$$u_\alpha = \bar{\psi}(x)\lambda_\alpha\psi(x) \tag{5d}$$

for $\alpha = 0, 1, ..., 8$, have been made. In the above, ψ stands for the three quark fields ψ_u, ψ_d, ψ_s and the λ_α for $\alpha = 0, 1, ..., 8$ are the well-known 3×3 Gell-Mann matrices acting on the components ψ_u, ψ_d, ψ_s of ψ.

The complete Hamiltonian of the standard model can now be written as

$$H(x) = H_0(x) + u(x)\,, \tag{6}$$

where only $u(x)$ breaks $SU(3) \otimes SU(3)$ chiral symmetry. Namely, defining the $SU(3) \otimes SU(3)$ vector and axial vector currents for $a = 1, ..., 8$ by $V^a_\mu = \frac{1}{2}\bar{\psi}\gamma_\mu\lambda_a\psi$ and $A^a_\mu = \frac{1}{2}\bar{\psi}\gamma_5\gamma_\mu\lambda_a\psi$, respectively, the corresponding charges $Q^a_V(x_0) = \int d^3x V^a_0(x)$ and $Q^a_A(x_0) = \int d^3x A^a_0(x)$ commute with $H_0(x)$,

$$[H_0(x), Q(x_0)] = 0\,, \tag{7}$$

such that

$$i[u(x), Q(x_0)] = \partial^\mu J_\mu(x) \tag{8}$$

for J_μ any one of these currents and $Q(x_0)$ the corresponding charge.

We remind the reader that equation (8) can be derived from the equation of motion of the standard model. More generally, equation (8) follows from equation (6) and (7) by use of the equal-time commutator (summation over $k = 1, 2, 3$ is understood)

$$i[H(x), J_0(y)]_{x_0 = y_0 = 0} = \partial^\mu J_\mu(x)\delta^{(3)}(\vec{x} - \vec{y}) + J_k(x)(\partial/\partial x_k)\delta^{(3)}(\vec{x} - \vec{y})\,, \tag{9}$$

which expresses (Genz and Katz 1971) the transformation properties of $J_0(y)$ under time translations by $H = \int d^3x H(x)$ and boosts by $M_{0k} = -\int d^3x x_k H(0, \vec{x})$.

Namely, by integrating (9) over d^3y due to (7) only $u(x)$ survives in the equal-time commutator of $H(x)$ with the charge $Q(x_0)$. Computing the equal-time commutators of the charges Q^a_V and Q^α_A (with Q^0_A included) with the scalar densities $u_\alpha(x)$, which we have already defined, and the pseudoscalar ones $v_\alpha = \bar{\psi}(x)\gamma_5\lambda_\alpha\psi(x)$ one finds for latin indices between 1 and 8 and greek indices between 0 and 8 the equal-time commutation relations [summation over double indices being understood; f and d are the well-known Clebsch–Gordan coefficients of $SU(3)$]

$$[Q_V^a, u_\beta] = i f_{a\beta c} u_c \,, \tag{10a}$$

$$[Q_V^a, v_\beta] = i f_{a\beta c} v_c \,, \tag{10b}$$

$$[Q_A^\alpha, u_\beta] = i d_{\alpha\beta\gamma} v_\gamma \,, \tag{10c}$$

$$[Q_A^\alpha, v_\beta] = -i d_{\alpha\beta\gamma} u_\gamma \,. \tag{10d}$$

For $\alpha \neq 0$ these are the defining relations of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU(3) \otimes SU(3)$.

As an immediate consequence, currents for which equation (7) holds are conserved in the limit of vanishing quark masses. Except for electromagnetic anomalies, which are not supposed to contribute to the VEV of the σ-commutators, Goldstone chiral symmetry breaking implies precisely this for the 16 currents of $SU(3) \otimes SU(3)$. The ninth $SU(3)$-neutral axial current A_μ^0 is however not assumed to have a vanishing divergence in this limit. Therefore we have stressed that our results which involve the divergence of this particular current presumably are of academic interest only.

The current divergencies turn out to be (summation once again being understood)

$$\partial^\mu V_\mu^a = (c_3 f_{a3c} + c_8 f_{a8c}) u_c \,, \tag{11a}$$

$$\partial^\mu A_\mu^\alpha = (c_0 d_{\alpha 0\gamma} + c_3 d_{\alpha 3\gamma} + c_8 d_{\alpha 8\gamma}) v_\gamma \,. \tag{11b}$$

It is now an easy exercise to compute the σ-commutators in terms of the u_α and v_α. The results for u_α imply the reconstruction theorem of the chiral $SU(3) \otimes SU(3)$ symmetry breaking Hamiltonian density which reads (Genz and Cornwell 1973)

$$u(x) = i C_R / 2([Q_V^a, \partial^\mu V_\mu^a] + [Q_A^a, \partial^\mu A_\mu^a]) \,, \tag{12}$$

where $C_R = 3/16$. Summation over $a = 1, ..., 8$ is understood. For the group-theoretically inclined reader we note that (12) with $C_R > 0$ follows for any u that belongs to an irreducible representation of $SU(3) \otimes SU(3)$ except the unit representation. The reason is that the Casimir operator $\frac{1}{2}(Q_V^a Q_V^a + Q_A^a Q_A^a)$ of $SU(3) \otimes SU(3)$ acting on u multiplies it with its (positive) eigenvalue C_R^{-1}.

Thus, from (1), we see that the VEV of $-u$ is non-negative (Genz and Cornwell 1973):

$$\langle -u(x) \rangle_0 \geq 0 \,, \tag{13}$$

for any u that belongs to an irreducible representation of $SU(3) \otimes SU(3)$. If the equals sign applies, the divergences of all partially conserved currents vanish.

Following Rausch (1982) we will exploit the VEV $\langle [Q_A^\alpha, \partial^\mu A_\mu^\beta] \rangle_0$ of the axial σ-terms and start with the remark that nothing additional could be gained by also considering the VEV of the others, i.e. $\langle [Q_V^a, \partial^\mu V_\mu^b] \rangle_0$, $\langle [Q_V^a, \partial^\mu A_\mu^\beta] \rangle_0$ and $\langle [Q_A^\alpha, \partial^\mu V_\mu^b] \rangle_0$, since these either vanish trivially due to current or parity conservation or since in the approximation in which we work the vector charges Q_V^a $(a = 1, ..., 8)$ annihilate the vacuum. This also implies that all scalar densities u_α except u_0 have vanishing VEVs since they can be represented as linear combinations of commutators of the type $[Q_V^a, u_b]$ such as e.g. $u_3 = -i[Q_V^1, u_2]$ and $u_8 = i\sqrt{3}[Q_V^1, u_2] - i2/\sqrt{3}[Q_V^4, u_5]$. The vanishing of the VEVs of u_1, u_2, u_4, u_5, u_6 and u_7 is also implied by the even stronger argument that—except for special values of c_3 and c_8—they are proportional to a vector current divergence, the VEV of which vanishes due to translation invariance. In any case, we may conclude

$$\langle u_\alpha \rangle_0 = \delta_{0\alpha} \langle u_0 \rangle_0 \,. \tag{14}$$

Returning to the VEV $-i\langle [Q_A^\alpha, \partial^\mu A_\mu^\beta] \rangle_0$ we will have to distinguish between the 8×8 matrix for α and β between 1 and 8 and the 9×9 matrix for the indices between 0 and 8. The 9×9 matrix can be written under our assumptions as

$$\begin{aligned}\sigma_{\alpha\beta} \equiv & -i\langle [Q_A^\alpha, \partial^\mu A_\mu^\beta] \rangle_0 \\ = & \sqrt{\tfrac{2}{3}} \langle -c_0 u_0 \rangle_0 [\delta_{\alpha\beta} + (c_3/c_0) d_{3\alpha\beta} + (c_8/c_0) d_{8\alpha\beta}] \,,\end{aligned} \tag{15}$$

which incidentally also shows that the matrix $\sigma_{\alpha\beta}$ is symmetric under exchange of α and β. Making the definitions

$$G = \tfrac{2}{3} \langle u \rangle_0 = \tfrac{2}{3} \langle c_0 u_0 \rangle_0 \,, \tag{16a}$$

$$A = c_8/c_0 \,, \tag{16b}$$

$$B = c_3/c_0 \,, \tag{16c}$$

we can write the non-vanishing elements of the symmetric and real matrix σ as

$$\sigma_{00} = -G \,, \tag{17a}$$

$$\sigma_{08} = \sigma_{80} = -GA \,, \tag{17b}$$

$$\sigma_{03} = \sigma_{30} = -GB \,, \tag{17c}$$

$$\sigma_{38} = \sigma_{83} = -GB/\sqrt{2} \,, \tag{17d}$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -G(1 + A/\sqrt{2}) \,, \tag{17e}$$

$$\sigma_{44} = \sigma_{55} = -G[1 + B\sqrt{\tfrac{3}{2}}/2 - A/(2\sqrt{2})] \,, \tag{17f}$$

$$\sigma_{66} = \sigma_{77} = -G[1 - B\sqrt{\tfrac{3}{2}}/2 - A/(2\sqrt{2})]\,, \tag{17g}$$

$$\sigma_{88} = \; - G(1 - A/\sqrt{2})\,. \tag{17h}$$

Effectively, we therefore are dealing with a 4×4 or 5×5 matrix rather than an 8×8 or 9×9 matrix, respectively. It should be noted that the fact that σ_{38} does not vanish immediately implies mixing of states with π^0 and η quantum numbers.

3. The Derivation

From (1) and the definition in (15) it follows that for z_m with $m = 0, 8, 3, 4, 6$ any five real numbers we have

$$z_m z_n \sigma_{mn} \geq 0\,, \tag{18}$$

where summation over m and n is understood. To exploit this condition of positivity of the real symmetric 4×4 or 5×5 matrix σ, we may apply (Rausch 1982) the Hausdorff criterion which states that an $N \times N$ matrix M_{ik} is positive if and only if for every k between 1 and N

$$\det \begin{pmatrix} M_{11} & ... & M_{1k} \\ ... & ... & ... \\ M_{k1} & ... & M_{kk} \end{pmatrix} \geq 0\,.$$

We start by exploiting partial conservation of the eight axial currents of $SU(3) \otimes SU(3)$. From $\sigma_{33} + \sigma_{44} + \sigma_{66}$ being non-negative we obtain $-G \geq 0$, i.e. under more restrictive assumptions once again (13). It is gratifying that this result is in agreement with σ_{00} in (17a) being non-negative. From $\sigma_{33} \geq 0$ we obtain the further result

$$A \geq -\sqrt{2}\,. \tag{19a}$$

That the 2×2 submatrix σ_{mn} for $m = 3, 8$ and $n = 3, 8$ is non-negative yields

$$2 \geq A^2 + B^2 \tag{19b}$$

and from considering σ_{44} and σ_{55} separately we get

$$B \geq a/\sqrt{3} - -2\sqrt{\tfrac{2}{3}}\,, \tag{19c}$$

$$2\sqrt{\tfrac{2}{3}} - A/\sqrt{3} \geq B\,. \tag{19d}$$

Taken together, these relations are evidently somewhat redundant.

Replacing Ac_0, Bc_0 and c_0 by the corresponding linear combinations of the quark masses and assuming that their sum is positive,

$$m_u + m_d + m_s > 0\,, \tag{20}$$

we obtain equations (2) as restrictions on these masses themselves. The changes that occur if $m_u + m_d + m_s < 0$ and the corresponding conclusions have already be stated in connection with (2).

Finally, if our assumptions are extended to also cover the ninth axial current A^0_μ, we may apply the Hurwitz criterion to the full 5×5 matrix σ. It suffices to consider the cases $k = 0, 8, 3$ with the results

$$-G \geq 0, \tag{21a}$$

$$1 - A/\sqrt{2} - A^2 \geq 0\,, \tag{21b}$$

$$(\sqrt{2a} - 1)[\tfrac{3}{2}B^2 - (1 + A/\sqrt{2})^2] \geq 0\,. \tag{21c}$$

As has already been said, (21a) already follows from our previous results. One easily sees that (21b) is equivalent to

$$1/\sqrt{2} \geq A \geq -\sqrt{2}\,, \tag{22a}$$

such that from (20c)

$$(1 + a/\sqrt{2})^2 \geq 3B^2\,, \tag{22b}$$

or equivalently

$$2 + A^2 + 2\sqrt{2}A \geq 3B^2\,. \tag{22c}$$

The relation $A \geq -\sqrt{2}$ has already been obtained in the above. Under the assumption that the sum of the three quark masses is positive, the additional relations are equivalent to

$$m_s \geq 0\,, \tag{23a}$$

$$m_u m_d \geq 0\,. \tag{23b}$$

Since we know that m_u and m_d cannot both be negative, they must be positive (or at least one of them must be zero). If the sum of the quark masses is negative, the greater-or-equal sign in (23a) is inverted, whereas (23b) remains unaltered. It then follows that that none of the three quark masses can be positive.

Our conclusions have already been noted in connection with (2). Under the assumption that none of the three quark masses vanishes, they can be stated as follows: (1) Partial conservation of the nine axial currents applied to the computation of the VEV of the σ-terms in the standard model implies that all three quark masses m_u, m_d and m_s have the same (positive or negative) sign. (2) From partial conservation of only the eight axial currents of $SU(3) \otimes SU(3)$ the inequalities in (2) follow. They imply for example that at least two of the three quark masses have the same sign as the sum $m_u + m_d + m_s$ of all of them.

The reader is reminded that our assumptions include the standard *vector* SU(3) symmetry of the vacuum.

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