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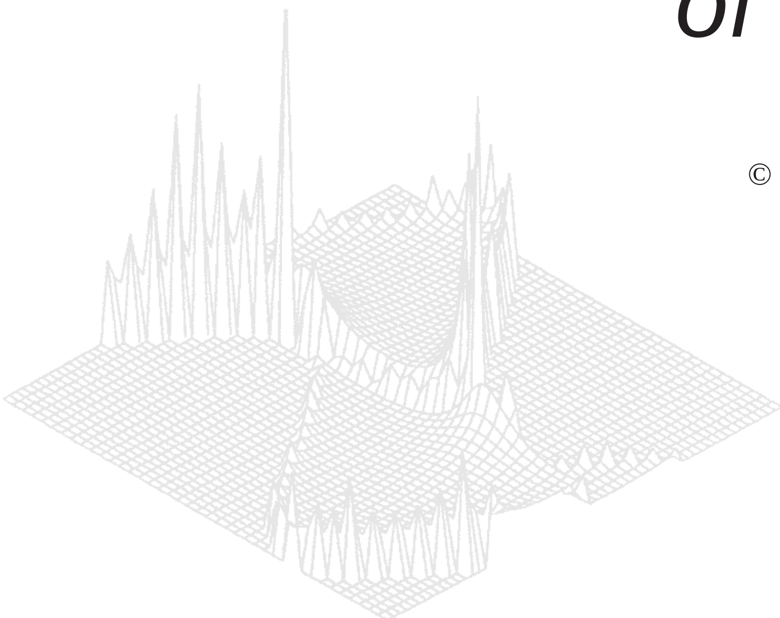

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Local Stability Analysis of Three-dimensional Magnetically Confined Plasmas

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Abstract

Local stability analysis of three-dimensional magnetically confined plasmas is presented using the energy principle. Fluid displacements with long parallel wavelengths by short perpendicular are retained. To lowest order in the mode fluteness the total energy density variation depends on the local magnetic shear, normal and geodesic magnetic curvatures, parallel current density, and magnetic field strength. The perpendicular fluid displacement is written in terms of a slowly-varying amplitude and a phase angle which varies rapidly in (or across) the magnetic surface. Assuming that the fluid displacement is quasi-incompressible, the solution for the phase angle is found in two limiting cases. Local stability analysis of stellarators with low or strong global magnetic shear is discussed.

1. Introduction

The plasma physics problems encountered in controlled thermonuclear fusion can be separated into four basic areas. These are (a) equilibrium, (b) stability, (c) transport and (d) heating. Issues (a)–(d) are intimately related but the solution of the self-consistent system is not tractable.

Where stability and equilibrium are concerned, the ideal magnetohydrodynamic (MHD) model is the simplest model for describing the interaction between a perfectly conducting fluid (the plasma) and a (confining) magnetic field. The basic requirement for the validity of the ideal MHD equations is that the bulk of the plasma be collisional. However, it is well-known that the ideal MHD limit is not valid for high-density, hot plasmas (Freidbreg 1982). However, experimental evidence suggests that the ideal MHD model provides a reasonable description of slow, macroscopic plasma phenomena.

Linearised MHD equations can be cast in a variational form known as the energy principle (Greene and Johnson 1968; Furth *et al.* 1966). The energy principle, valid for arbitrary magnetic geometry, gives exact information about instability thresholds and estimates the mode eigenfrequencies (growth rates). Because of its inherent mathematical simplicity, the energy principle offers an attractive method to consider plasma stability of fully three-dimensional plasma geometries (such as stellarators and tokamaks with coil ripple effects).

Ideal MHD instabilities can be driven by the parallel current density (current-driven modes) or by the perpendicular pressure gradient (pressure-driven modes). One of the most dangerous current-driven modes, at least for the tokamak

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configuration, is the kink instability (Schneider and Bateman 1974; Todd *et al.* 1977). For currentless stellarators this type of instability is usually absent during plasma discharges. Pressure-driven modes include the interchange mode (Bateman 1978) and the ballooning mode (Coppi 1977). Ballooning modes set a threshold β_c [$\beta \equiv$ plasma kinetic pressure/magnetic energy]. Below the threshold β_c, that is for $\beta < \beta_c$, some slowly-growing modes might survive.

In this paper we consider the energy principle for three-dimensional plasma geometries with $\beta < \beta_c$. The potential energy variation is derived by using an expansion in terms of a smallness parameter $\Delta \equiv k_{||}/k_{\perp}$. Here $2\pi/k_{||}$ and $2\pi/k_{\perp}$ are the parallel and perpendicular wavelengths of the fluid displacement respectively. Assuming that the fluid displacement satisfies the quasi-incompressibility condition, the potential energy variation is shown to depend on magnetic field key attributes such as the local magnetic shear, the normal and geodesic curvatures, the magnetic field strength and the parallel current density. In the low and strong global shear limits, the phase angle of the fluid displacement can be obtained by an expansion method.

The paper is organised as follows. In Section 2, the energy principle is considered for fluid displacement with $\Delta \equiv k_{||}/k_{\perp} \ll 1$. Analytical expressions for the local magnetic shear, the normal and geodesic curvature are presented. In Section 3, the incompressibility condition (assumed in Section 2) is derived for arbitrary three-dimensional plasma geometry. The phase angle of the fluid displacement is found in the low and strong global shear limits. Conclusions are presented in Section 4.

2. The Energy Principle

In the context of ideal MHD theory, viscosity effects, heat flow, ohmic dissipation and resistivity are neglected. The plasma is seen as a dense, highly conducting medium and the displacement current is neglected (Greene *et al.* 1962). Plasma fluid elements move with the magnetic field (Newcomb 1958) (frozen-in condition). Furthermore, steady-state plasma flows are neglected (Greene *et al.* 1962). The macroscopic plasma stability can be studied by introducing small perturbations in the equilibrium configuration (Berstein *et al.* 1958). If ξ measures the displacement of a fluid element from its equilibrium position, a variational principle can be constructed (Berstein *et al.* 1958; Greene *et al.* 1962). The plasma potential energy variation due to the perturbation ξ is conveniently written as follows (Greene and Johnson 1968; Furth *et al.* 1966):

$$\delta W = \delta W_1 + \delta W_2 + \delta W_3 + \delta W_4 \,, \tag{1}$$

where

$$\delta W_1 \equiv \frac{1}{2\mu_0} \int d^3x \left[\delta \mathbf{B}_{\perp}{}^2 + \left(\delta \mathbf{B}_{||} - \mu_0 \frac{\xi \cdot \nabla}{B^2} \right)^2 \right] , \tag{2}$$

$$\delta W_2 \equiv -\tfrac{1}{2} \int d^3x \frac{j_{||}}{B} \; (\xi \times \mathbf{B}) \cdot \nabla \times (\xi \times \mathbf{B}) \,, \tag{3}$$

$$\delta W_3 \equiv \frac{1}{2} \int d^3x \gamma p (\nabla \xi)^2 \,, \tag{4}$$

$$\delta W_4 \equiv \int d^3x (\xi \cdot \nabla p)(\xi \cdot \kappa) \,. \tag{5}$$

Terms of order ξ^3 and higher have been neglected and linear terms vanish because we perturb around equilibrium; d^3x is a plasma volume element and integration is carried out over the entire plasma volume. The plasma is assumed to be surrounded by a conducting wall at which the normal component of the fluid displacement vanishes. Here $\mathbf{B}$ and $\delta\mathbf{B} \equiv \nabla \times (\xi \times \mathbf{B})$ are the equilibrium and perturbed magnetic fields respectively; subscripts '$||$' and '$\perp$' refer to the directions parallel and perpendicular to the equilibrium magnetic field respectively; γ is the adiabatic index and $\gamma = \frac{5}{3}$ for a system with degrees of freedom; p and $j_{||}$ are the equilibrium plasma pressure and parallel current density respectively; and κ is the magnetic curvature.

The first term in δW_1 represents the energy required to bend the magnetic field line while the second term corresponds to the energy necessary to compress the magnetic field. The second term, δW_2, is the free energy coming from the parallel current density and is responsible for kink instabilities (Schneider and Bateman 1974; Todd *et al.* 1977). The third term in equation (1) represents the energy required to compress the plasma and it is the main source of potential energy for sound waves. The remaining contribution to the potential energy variation δW_4, proportional to the perpendicular current density, is responsible for interchange (Bateman 1978) and ballooning (Coppi 1977) instabilities. This term is related to the tension of magnetic field lines. This tension leads to a force that is proportional to the curvature of the field lines and to the magnitude of the tension which, in turn, is proportional to B^2.

The positive-definite contributions δW_1 and δW_3 are always stabilising but the current-driven terms, δW_2 and δW_4, can be positive or negative and can lead to an instability.

In three-dimensional plasma geometry it is convenient to perform analytical and numerical calculations in curvilinear coordinates. At equilibrium, magnetic surfaces are assumed to consist of a family of nested tori. In this case, the plasma pressure is a flux surface quantity, i.e. p is uniform on a given magnetic surface. The existence of magnetic surfaces ensures that the enclosed poloidal magnetic flux ($\equiv 2\pi\psi$) and the toroidal magnetic flux ($\equiv 2\pi\Phi$) are flux surface quantities. There are mild advantages to using a normalised radial label. The normalised radial label is denoted $s \equiv \Phi/\Phi_a$ (where $2\pi\Phi_a$ is the toroidal magnetic flux at the plasma edge) and runs from 0 (at the magnetic axis) to 1 (at the plasma edge). The radial label s is supplemented with curvilinear (or magnetic) poloidal and toroidal angles θ and ζ respectively.

However, in order to get physical insight, intermediate analytical calculations are carried out using local coordinates. A local vector system attached to the magnetic field lines can be defined:

$$\mathbf{e}_{||} \equiv \mathbf{B}/B \tag{6}$$

is a unit vector parallel to the magnetic field direction. Since a magnetic surface is defined as $s = \text{const}$, so that ∇s is normal to the magnetic surface, one can introduce a unit normal vector:

$$\mathbf{n} \equiv \nabla s/(\nabla s \,.\, \nabla s)^{\frac{1}{2}} \,. \tag{7}$$

Finally,

$$\mathbf{b} \equiv \mathbf{e}_{||} \times \mathbf{n} \tag{8}$$

is the so-called unit binormal vector. By construction, $\mathbf{b}$ is perpendicular to both the magnetic field and the normal to the magnetic surface. The orthonormal basis vectors set $\{\mathbf{e}_{||}, \mathbf{n}, \mathbf{b}\}$ is sometimes called the moving Frenet–Serret trihedron (D'haeseleer *et al.* 1983). Each basis vector depends on all three magnetic coordinates $\{s, \theta, \zeta\}$ and their explicit forms are computed from the magnetic field components obtained from an equilibrium code. Any vector $\mathbf{F}$ can be decomposed onto the moving Frenet–Serret trihedron: $\mathbf{F} = F_{||}\,\mathbf{e}_{||} + F_{\mathrm{n}}\,\mathbf{n} + F_{\mathrm{b}}\,\mathbf{b}$, where $F_{||} \equiv \mathbf{e}_{||} \,.\, \mathbf{F}$, $F_{\mathrm{n}} \equiv \mathbf{n} \,.\, \mathbf{F}$ and $F_{\mathrm{b}} \equiv \mathbf{b} \,.\, \mathbf{F}$ are the parallel, normal and binormal (or geodesic) components of the vector $\mathbf{F}$ respectively. Each vector quantity that enters the plasma potential energy variation (1) can be decomposed onto the moving Frenet–Serret trihedron. In particular the perturbed magnetic field can be written in the following form:

$$\begin{aligned}\delta\mathbf{B} = &\left\{B^2(\mathbf{e}_{||} \,.\, \nabla)(\xi_{||}/B) - B(\nabla \,.\, \xi + 2\xi \,.\, \kappa) + \frac{\mu_0}{B}\dot{p}\sqrt{g^{ss}}\xi_n\right\}\mathbf{e}_{||} \\ &+ \left\{\frac{B}{\sqrt{g^{ss}}}\;(\mathbf{e}_{||} \,.\, \nabla)(\sqrt{g^{ss}}\xi_n)\right\}\mathbf{n} \\ &+ \left\{\sqrt{g^{ss}}(\mathbf{e}_{||} \,.\, \nabla)\left(\frac{B}{\sqrt{g^{ss}}}\;\xi_b\right) - \mathcal{S}(\nabla\psi \,.\, \nabla\psi)\xi_n\right\}\mathbf{b} \,.\end{aligned} \tag{9}$$

Here $\kappa \equiv (\mathbf{e}_{||} \,.\, \nabla)\mathbf{e}_{||}$ is the magnetic curvature, $g^{ss} \equiv \nabla s \,.\, \nabla s$ is a metric element and $2\pi\psi(s)$ is the enclosed poloidal flux. A dot denotes a derivative with respect to s. Further, $\mathcal{S}$ is the local magnetic shear (Dewar *et al.* 1984; Greene and Chance 1981). In equation (9) we note that the parallel component of the fluid displacement does not enter the perpendicular component of the perturbed magnetic field.

Following Dewar *et al.* (1984), the local magnetic shear (LMS) is defined as

$$\mathcal{S} \equiv -\mathbf{s} \,.\, \nabla \times \mathbf{s} \,, \tag{10}$$

where $\mathbf{s} \equiv (\mathbf{B} \times \nabla\psi)/(\nabla\psi \,.\, \nabla\psi)$ is a vector lying in the magnetic surface and its direction is perpendicular to the equilibrium magnetic field direction. As noted by Ware (1965), the last term in equation (9) is the amount a field line must be stretched if it is to exactly replace a neighbouring field line in the course of the perturbation.

Since the enclosed poloidal flux is a flux surface quantity, this implies that the vector $\nabla\psi$ is directed along the normal to the magnetic surface. In view of equation (8), it follows that the vector $\mathbf{s}$ is directed along the binormal direction. After straightforward algebra, the LMS can be rewritten in the following form:

$$\mathcal{S} = -P\mathbf{b} \cdot (\nabla \times \mathbf{b}) \,, \tag{11}$$

where $P \equiv (B/\dot{\psi}\sqrt{g}^{ss})^2 > 0$ is a positive definite quantity. Therefore, the LMS is proportional to the projection of the rotational of the binormal vector onto itself. After straightforward algebra, we get the LMS in curvilinear coordinates:

$$\mathcal{S} = P\left\{ b_s\left(\frac{\partial b_\theta}{\partial \zeta} - \frac{\partial b_\zeta}{\partial \theta}\right) + b_\theta\left(\frac{\partial b_\zeta}{\partial s} - \frac{\partial b_s}{\partial \zeta}\right) + b_\zeta\left(\frac{\partial b_s}{\partial \theta} - \frac{\partial b_\theta}{\partial s}\right)\right\}, \tag{12}$$

where the covariant components of the binormal vector are

$$\begin{aligned} b_s &\equiv \mathbf{b}\cdot\mathbf{e}_s = \mathcal{J}(B^\theta g^{s\zeta} - B^\zeta g^{s\theta})/(B\sqrt{g^{ss}}) \,, \\ b_\theta &\equiv \mathbf{b}\cdot\mathbf{e}_\theta = \mathcal{J}B^\zeta\sqrt{g^{ss}}/B \,, \\ b_\zeta &\equiv \mathbf{b}\cdot\mathbf{e}_\zeta = -\mathcal{J}B^\theta\sqrt{g^{ss}}/B \,. \end{aligned} \tag{13}$$

MHD stability, plasma transport and microinstability theories have shown that the *surface average* of the local magnetic shear is a key attribute of a confinement device. The local magnetic shear is written as

$$\mathcal{S} = \widehat{S} + \mathcal{R} \,, \tag{14}$$

where the global magnetic shear $\widehat{S}$ is a surface average quantity,

$$\widehat{S}(s) \equiv A(s)^{-1}\int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \mathcal{J}\mathcal{S}(s,\theta,\zeta) \,, \tag{15}$$

where $\mathcal{J}$ is the Jacobian of the transformation, $\mathcal{J}^{-1} \equiv \nabla s \cdot (\nabla\theta \times \nabla\zeta)$, and

$$A(s) \equiv \int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta \mathcal{J} \tag{16}$$

is a flux surface quantity proportional to the area of the magnetic surface. Here $\mathcal{R}$ is the so-called residual shear Dewar *et al.* (1984). By construction, the surface average of the residual shear identically vanishes. The existence of global shear eliminates the possibility of a plasma perturbation for which all stabilising terms vanish (Greene and Johnson 1968).

A second key attribute of a confinement device is the curvature of the magnetic field lines. Projecting the gradient operator onto the contravariant basis vectors, the magnetic curvature reads

$$\kappa = (b^\theta\partial/\partial\theta + b^\zeta\partial/\partial\zeta)\mathbf{e}_{||} \,, \tag{17}$$

where $b^\theta \equiv B^\theta/B$ and $b^\zeta \equiv B^\zeta/B$. After straightforward algebra, the magnetic field curvature in arbitrary curvilinear coordinates assumes the following form:

$$\kappa = \left[b^\theta \frac{\partial b^\theta}{\partial \theta} + b^\zeta \frac{\partial b^\theta}{\partial \zeta}\right]\mathbf{e}_\theta + \left[b^\theta \frac{\partial b^\zeta}{\partial \theta} + b^\zeta \frac{\partial b^\zeta}{\partial \zeta}\right]\mathbf{e}_\zeta$$
$$+ 2b^\theta b^\zeta \mathbf{G}_{\theta\zeta} + (b^\theta)^2\mathbf{G}_{\theta\theta} + (b^\zeta)^2\mathbf{G}_{\zeta\zeta}\,. \tag{18}$$

Here, $\mathbf{e}_\theta \equiv \partial\mathbf{r}/\partial\theta$ and $\mathbf{e}_\zeta \equiv \partial\mathbf{r}/\partial\zeta$, where $\mathbf{r}$ is the local position vector on a given magnetic surface. The last three terms in equation (18) include the curvature effects

$$\mathbf{G}_{\alpha\beta} \equiv \frac{\partial^2\mathbf{r}}{\partial\alpha\partial\beta}\,, \tag{19}$$

for $(\alpha,\beta) = \{\theta,\zeta\}$. It is customary to decompose the magnetic curvature into two components. The component of the magnetic curvature along $\nabla\psi$ is the so-called normal curvature. The component of κ lying *in* the magnetic surface is the geodesic curvature. As we shall see below, the normal and geodesic components of the magnetic curvature enter in concert in the final form of the potential energy variation.

Dotting the magnetic curvature vector with the unit normal vector (7) leads to the normal curvature:

$$\kappa_{\mathrm{N}} = 2b^\theta b^\zeta \mathbf{e}^s \,.\, \mathbf{G}_{\theta\zeta} + (b^\theta)^2\mathbf{e}^s \,.\, \mathbf{G}_{\theta\theta} + (b^\zeta)^2\mathbf{e}^s \,.\, \mathbf{G}_{\zeta\zeta}\,, \tag{20}$$

whereas, dotting equation (18) with the unit binormal (geodesic) vector, the geodesic curvature reads

$$\kappa_{\mathrm{G}} = \frac{1}{\mathcal{J}\sqrt{g}^{ss}}\;[b_\zeta\kappa_\theta - b_\theta\kappa_\zeta]\,, \tag{21}$$

where

$$\kappa_\theta \equiv \left[b^\theta \frac{\partial b^\theta}{\partial \theta} + b^\zeta \frac{\partial b^\theta}{\partial \zeta}\;\right]g_{\theta\theta} + \left[b^\theta \frac{\partial b^\zeta}{\partial \theta} + b^\zeta \frac{\partial b^\zeta}{\partial \zeta}\;\right]g_{\theta\zeta}$$
$$+ 2b^\theta b^\zeta \mathbf{e}_\theta \,.\, \mathbf{G}_{\theta\zeta} + (b^\theta)^2\mathbf{e}_\theta \,.\, \mathbf{G}_{\theta\theta} + (b^\zeta)^2\mathbf{e}_\theta \,.\, \mathbf{G}_{\zeta\zeta}\,, \tag{22}$$

$$\kappa_\zeta \equiv \left[b^\theta \frac{\partial b^\theta}{\partial \theta} + b^\zeta \frac{\partial b^\theta}{\partial \zeta}\;\right]g_{\theta\zeta} + \left[b^\theta \frac{\partial b^\zeta}{\partial \theta} + b^\zeta \frac{\partial b^\zeta}{\partial \zeta}\;\right]g_{\zeta\zeta}$$
$$+ 2b^\theta b^\zeta \mathbf{e}_\zeta \,.\, \mathbf{G}_{\theta\zeta} + (b^\theta)^2\mathbf{e}_\zeta \,.\, \mathbf{G}_{\theta\theta} + (b^\zeta)^2\mathbf{e}_\zeta \,.\, \mathbf{G}_{\zeta\zeta}\,. \tag{23}$$

For the remainder of the paper we shall restrict our attention to perturbations that are localised near a line of force. The fluid displacement is assumed to be slowly varying along the equilibrium magnetic field direction. Typical perpendicular wavelengths for perturbations are assumed to be small but much larger than the ion gyro-radius ($k_\perp\rho_i \ll 1$) so that the use of a fluid model remains valid. In view of this strong anisotropy, it is convenient to introduce a smallness parameter

$$\Delta \equiv k_{||}/k_{\perp} \, , \tag{24}$$

where $k_{||} \sim \nabla_{||}\xi/\xi$ and $k_{\perp} \sim \nabla_{\perp}\xi/\xi$ are the effective parallel and perpendicular wavelengths of the fluid displacement respectively. The parameter Δ measures the degree of the mode 'fluteness'. The most pessimistic perturbations are characterised by large parallel wavelength because the stabilising effect associated with magnetic field line bending is significantly reduced. In this paper, Δ is used as an ordering parameter and it is assumed to be small but finite.

Equilibrium quantities are assumed to slowly vary in space with a characteristic scalelength L. We allow a long parallel wavelength for perturbations so that $k_{||} \sim 1/L$. The fluid motion is assumed to be quasi-incompressible, $\nabla\,.\,\xi \sim 0$ (see next section). To lowest order in the mode fluteness the normal component of the perturbed magnetic field vanishes. This can be easily derived from equation (9) by setting $\mathbf{e}_{||}\,.\,\nabla \equiv \nabla_{||} \mapsto 0$. The perturbed magnetic field reads

$$\frac{\delta\mathbf{B}}{B} = -\left\{2\xi\,.\,\kappa + \frac{\beta}{2}\frac{\sqrt{g^{ss}}}{\lambda(s)}\xi_{\mathrm{n}}\right\}\mathbf{e}_{||} + \left\{\mathcal{S}\frac{\nabla\psi\,.\,\nabla\psi}{B^2}\xi_{\mathrm{n}}\right\}\mathbf{b}\,. \tag{25}$$

Here $\lambda^{-1}(s) \equiv -d\ln p(s)/ds > 0$ is the normalised plasma pressure gradient scalelength and $\beta \equiv 2\mu_0 p(s)/B^2$ is the ratio of the plasma potential energy to the magnetic energy. The fluid displacement compresses the magnetic field lines [first term in equation (25)] and bends them *in* the magnetic surface [second term in equation (25)]. Magnetic curvature and pressure gradient are responsible for the compressional term. The change in the direction of the total magnetic field due to the fluid displacement is related to the amount of shear (Greene and Johnson 1968). The parallel component of the perturbed magnetic field is

$$\frac{\delta\mathbf{B}_{||}}{B} = -\left[2\xi\,.\,\kappa + \frac{\beta}{2}\;\frac{\sqrt{g^{ss}}}{\lambda(s)}\xi_{\mathrm{n}}\right]\mathbf{e}_{||}\,. \tag{26}$$

The component of the *perturbed* magnetic field which is orthogonal to the *confining* magnetic field, correct to lowest order in Δ, is

$$\frac{\delta\mathbf{B}_{\perp}}{B} = \mathcal{S}\frac{\nabla\psi\,.\,\nabla\psi}{B^2}\xi_{\mathrm{n}}\mathbf{b}\,. \tag{27}$$

The normal component of the perturbed magnetic field is $\mathcal{O}(\Delta)$. The term involving the magnetic curvature can be simplified as follows:

$$\begin{aligned}\kappa\,.\,\xi &= (\kappa_{\mathrm{N}}\mathbf{n} + \kappa_{\mathrm{G}}\mathbf{b})\,.\,(\xi_{||}\mathbf{e}_{||} + \xi_{\mathrm{b}}\mathbf{b} + \xi_{\mathrm{n}}\mathbf{n}) \\ &= \kappa_{\mathrm{N}}\xi_{\mathrm{n}} + \kappa_{\mathrm{G}}\xi_{\mathrm{b}}\,,\end{aligned} \tag{28}$$

where we have made use of the fact that the parallel component of the magnetic field curvature identically vanishes, $\mathbf{e}_{||}\,.\,\kappa = -\mathbf{e}_{||}\,.\,[\mathbf{e}_{||}\times(\nabla\times\mathbf{e}_{||})] \equiv 0$. Substituting equation (28) in (26) leads to

$$\frac{\delta\mathbf{B}_{||}}{B} - \left[2(\kappa_{\mathrm{N}}\xi_{\mathrm{n}} + \kappa_{\mathrm{G}}\xi_{\mathrm{b}}) + \frac{\beta}{2}\ \frac{\sqrt{g^{ss}}}{\lambda(s)}\xi_{\mathrm{n}}\right]\mathbf{e}_{||}\,. \tag{29}$$

Using equations (29) and (27) the field line bending contribution (δW_1) becomes

$$\delta W_1 = \frac{1}{2\mu_0}\ \int d^3xB^2\left\{\mathcal{S}^2\frac{(\nabla\psi\,.\,\nabla\psi)^2}{B^4}\ \xi_{\mathrm{n}}^2 + 4(\kappa_{\mathrm{N}}\xi_{\mathrm{n}} + \kappa_{\mathrm{G}}\xi_{\mathrm{b}})^2\right\}, \tag{30}$$

where we have noted the cancellation of finite-β terms. The integrand of the 'kink term', δW_2, can be simplified as follows:

$$\begin{aligned}(\xi\times\mathbf{B})\,.\,\nabla\times(\xi\times\mathbf{B}) &= \frac{\delta\mathbf{B}}{B}\,.\,[(\xi_{||}\mathbf{e}_{||} + \xi_{\mathrm{n}}\mathbf{n} + \xi_{\mathrm{b}}\mathbf{b})\times\mathbf{e}_{||}]\\ &= \frac{\delta\mathbf{B}}{B}\,.\,(\xi_{\mathrm{b}}\mathbf{n} - \xi_{\mathrm{n}}\mathbf{b}) = -\mathcal{S}\frac{\nabla\psi\,.\,\nabla\psi}{B^2}\ \xi_{\mathrm{n}}^2\,,\end{aligned} \tag{31}$$

since $\delta B_n = \delta\mathbf{B}\,.\,\mathbf{n} = 0$ to lowest order in Δ. We have also made use of equation (27). The term responsible for ballooning instabilities, δW_4, can be written

$$\begin{aligned}\delta W_4 &= \int d^3x(\xi\,.\,\nabla p)(\xi\,.\,\kappa)\\ &= \int d^3x\frac{dp}{ds}\ (\xi\,.\,\nabla s)(\xi_{\mathrm{n}}\kappa_{\mathrm{N}} + \xi_{\mathrm{b}}\kappa_{\mathrm{G}})\\ &= \int d^3x\frac{dp}{ds}\ \sqrt{g^{ss}}\xi_{\mathrm{n}}(\xi_{\mathrm{n}}\kappa_{\mathrm{N}} + \xi_{\mathrm{b}}\kappa_{\mathrm{G}})\,.\end{aligned} \tag{32}$$

Collecting equations (30)–(32), the plasma potential energy variation becomes, to lowest order in the mode fluteness,

$$\begin{aligned}\rho = \rho_{\mathrm{mag}}\{&4(\xi_{\mathrm{n}}\kappa_{\mathrm{N}} + \xi_{\mathrm{b}}\kappa_{\mathrm{G}})^2 + \mathcal{S}^2Q^2\xi_{\mathrm{n}}^2\\ &- \frac{\xi_{\mathrm{n}}}{\lambda(s)}\beta\sqrt{g^{ss}}(\xi_{\mathrm{n}}\kappa_{\mathrm{N}} + \xi_{\mathrm{b}}\kappa_{\mathrm{G}}) + \sigma\mathcal{S}Q\xi_{\mathrm{n}}^2\}\,,\end{aligned} \tag{33}$$

where $\rho_{\mathrm{mag}} \equiv B^2/2\mu_0$ is the equilibrium magnetic energy density and $\sigma \equiv \mu_0 j_{||}/B$. Here $Q \equiv \nabla\psi\,.\,\nabla\psi/B^2$ is a positive definite quantity. The first two terms in equation (33) are always stabilising. The finite-β term can be either stabilising ($\kappa_{\mathrm{N}} < 0$) or destabilising ($\kappa_{\mathrm{N}} > 0$). The LMS enters the energy density as a stabilising effect through field line bending [second term in equation (33)] but can be either stabilising or destabilising through the free energy of the parallel current density [last term in equation (33)]. If the normal component of the fluid displacement is directed outwards from a magnetic surface ($\xi_{\mathrm{n}} > 0$), the normal magnetic curvature can be stabilising [first term in equation (33)] or can drive instability [third term in equation (33)].

The first two terms in (33) represent the stabilising effect of the magnetic field line bending. The local magnetic shear enters as a positive definite quantity and it is therefore stabilising. Using equation (14), the stabilising contribution of the LMS scales like

$$\mathcal{S}^2 = \widehat{S}^2 + \mathcal{R}^2 + 2\widehat{S}\mathcal{R}\,. \tag{34}$$

In order to get physical insight, we consider four distinct configurations: the conventional and reversed-shear tokamak configurations, the low-shear stellarator configuration and the high-shear stellarator.

For the conventional tokamak configuration, the global shear dominates so that $\mathcal{S}^2 \approx \widehat{S}^2 > 0$. This situation is favourable for plasma stability. For low-β tokamak plasmas ($\beta \sim \epsilon^2$ where $\epsilon \equiv a/R$ is the inverse aspect ratio), an analytical form for the LMS can be found by expanding the LMS in powers of the smallness parameter ϵ. The LMS for low-β tokamak plasmas reads, correct to $\mathcal{O}(\epsilon)$ (Lewandowski and Persson 1995, 1996):

$$\mathcal{S} = \widehat{S} - \frac{r}{R_0}\;\cos\theta\left[1 + R_0\frac{d^2\Delta_\star}{dr^2}\; + \frac{R_0}{r}\;\Delta_\star(1 - 2\widehat{S})\right], \tag{35}$$

where $\widehat{S} = r(dq/dr)/q$ is the global shear, r is the local minor radius and θ is the local poloidal angle measured from the plasma outboard. Here $\Delta_\star$ is the Shafranov (1963, 1966) shift. The second term on the right-hand side of (35) is the residual shear and it is negative around $\theta \approx 0$, where the normal curvature is strongly destabilising. For the reversed-shear configuration, $\widehat{S} < 0$, the residual shear becomes more negative so that the second term in (34) becomes more positive. More importantly, the last term in (34) is negative for the conventional configuration but becomes positive for the reversed-shear configuration. Therefore, a negative global shear increases the stabilising contribution of the field line bending term. This can partially explain the improvement of plasma confinement and stability observed in recent reversed-shear configurations in TPX (Kessel *et al.* 1994), TFTR (Rice *et al.* 1996; Phillips *et al.* 1996; Batha *et al.* 1996), JET (Hugon *et al.* 1992) and DIII-D (Strait 1994; Lazarus *et al.* 1991).

For stellarators with low global shear, such as H1-NF (Hamberger *et al.* 1990; Gardner and Blackwell 1992), TJ-II (Aledaldre *et al.* 1990) and WII-AS (Grieger *et al.* 1985), spatial variation of the LMS *in* the magnetic surface dominates: $\mathcal{S} \approx \mathcal{R}$. In this case, the region of the magnetic surface with small residual shear and unfavourable curvature is expected to be unstable.

For the conventional stellarator, such as LHD, the global shear is negative in the bulk of the plasma, vanishes at some radial position $s = s_\star$ and becomes positive for $s > s_\star$. In the region of negative global shear, the last term in equation (34) is stabilising (destabilising) if the residual shear is negative (positive). In general the sign and magnitude of $\mathcal{R}$ have to be determined numerically.

3. The Incompressibility Condition

In the previous section we have assumed that the fluid displacement is quasi-incompressible. For localised perturbations, gradients perpendicular to the line of force must be large and compressive terms in the energy density will dominate unless $\nabla\,.\,\xi \sim 0$. The condition $\nabla\,.\,\xi \sim 0$ imposes constraints on the form of ξ.

The most pessimistic perturbation is obtained by minimising δW with respect to $\xi_{||}$, provided the parallel gradient operator is nonsingular (Freidbreg 1982). If the Wentzel–Kramers–Brillouin (WKB) representation is used to formulate the form of ξ, the total energy density yields the general ballooning mode energy principle (Dewar and Glasser 1983). The resulting high-n ballooning equation is essentially one-dimensional (Dewar and Glasser 1983) and sets a threshold $\beta_{\rm c}$ that the plasma can sustain before becoming unstable. However, for $\beta < \beta_{\rm c}$, one has to retain the two-dimensional nature of the fluid displacement.

To lowest order in the mode fluteness, the parallel component of the fluid displacement does not enter the expression for the energy density. Therefore, without loss of generality, the fluid displacement can be written

$$\boldsymbol{\xi} = \widehat{\xi}(\cos\varphi \mathbf{n} + \sin\varphi \mathbf{b})\,, \tag{36}$$

where $\varphi \equiv \tan^{-1}(\boldsymbol{\xi}\,.\,\mathbf{b}/\boldsymbol{\xi}\,.\,\mathbf{n})$ is the angle between the binormal and normal components of the fluid displacement. The amplitude $\widehat{\xi}$ is assumed to be a slowly-varying function of the position. For the strong global shear case, the phase angle φ is allowed to vary rapidly *across* the magnetic surface. For the low global shear case, the phase angle varies rapidly *in* the magnetic surface. Solutions for φ are presented in the next section for the low and strong global shear limits.

Substituting representation (36) for the fluid displacement element in the energy density (33), the plasma potential energy variation is conveniently written as follows:

$$\delta W = \int_0^1 \langle\rho_{\rm mag}\rangle ds \int_0^{2\pi} d\theta \int_0^{2\pi} d\zeta F \widehat{\xi}^2\,, \tag{37}$$

where the function F is given by

$$\begin{aligned} F \equiv \mathcal{J}\frac{B^2}{\langle B^2\rangle}\cos^2\varphi\{4\,[\kappa_{\rm N} + \tan\varphi\kappa_{\rm G}]^2 + \mathcal{S}^2 Q^2 \\ - \frac{\beta}{\lambda(s)}\,\sqrt{g^{ss}}[\kappa_{\rm N} + \tan\varphi\kappa_{\rm G}] + \sigma\mathcal{S}Q\}\,. \end{aligned} \tag{38}$$

As before $\langle$ ... $\rangle$ denotes an average over the magnetic surface. Substituting representation (36) into the incompressibility condition and assuming $\nabla_\perp\widehat{\xi} \ll \nabla_\perp\varphi$ leads to the following nonlinear differential equation for the phase angle:

$$(\nabla\,.\,\mathbf{n} + \mathbf{b}\,.\,\nabla\varphi)\cos\varphi + (\nabla\,.\,\mathbf{b} - \mathbf{n}\,.\,\nabla\varphi)\sin\varphi = 0\,. \tag{39}$$

There is an infinite number of periodic solutions for φ satisfying equation (39). However, approximate solutions can be found in the low and strong global shear regimes. Using equation (10) we note that the global magnetic shear scales like $\bar{R}^{-3}$, where $\bar{R}$ is the average major radius of the configuration. The characteristic shear length is $L_{\mathcal{S}} \equiv |\widehat{S}|^{-1/3}$. For configurations with vanishing global shear $L_{\mathcal{S}}$ is readily infinite.

(3a) Low Global Shear Case

For a configuration with low global shear one can introduce a smallness parameter $\epsilon_{ls} \equiv \bar{R}/L_{\mathcal{S}} \ll 1$. The phase angle for the fluid displacement is expanded in terms of ϵ_{ls},

$$\varphi = \varphi^{(0)} + \varphi^{(1)}(\theta, \zeta) + \varphi^{(2)}(s, \theta, \zeta) + \dots , \tag{40}$$

where subscripts indicate the corresponding order in ϵ_{ls}. Substituting representation (40) in the differential equation (39), the lowest order contribution satisfies the relation

$$\tan \varphi^{(0)} = -\frac{\langle \nabla \cdot \mathbf{n} \rangle}{\langle \nabla \cdot \mathbf{b} \rangle} . \tag{41}$$

The solution to next order satisfies the following *linear* differential equation:

$$Q_\theta \frac{\partial \varphi^{(1)}}{\partial \zeta} - Q_\zeta \frac{\partial \varphi^{(1)}}{\partial \theta} - \tan(\varphi^{(0)} + \varphi^{(1)}) \nabla \cdot \mathbf{b} - \nabla \cdot \mathbf{n} = 0 , \tag{42}$$

where we have introduced $Q_\theta \equiv B_\theta / \mathcal{J} B \sqrt{g^{ss}}$ and $Q_\zeta \equiv B_\zeta / \mathcal{J} B \sqrt{g^{ss}}$. Equation (42) can be solved numerically with periodic boundary conditions in the poloidal and toroidal angles.

(3b) Strong Global Shear Case

When the shear length is much shorter than the plasma major radius, one can introduce a small parameter $\epsilon_{hs} \equiv L_{\mathcal{S}}/\bar{R}$ which is the reciprocal of the smallness parameter introduce in the previous case. In the strong shear case, the lowest order solution for the phase angle is a flux surface quantity. To lowest order, we get

$$\tan \varphi^{(0)} \frac{d\varphi^{(0)}}{ds} = H(s) , \tag{43}$$

where

$$H(s) \equiv \langle \nabla \cdot \mathbf{n} + \nabla \cdot \mathbf{b} \rangle \langle \sqrt{g^{ss}} \rangle , \tag{44}$$

is a flux surface quantity. The normal component of the fluid displacement at the conducting wall must vanish, $\xi \cdot \mathbf{n} = 0$. Therefore equation (43) has to be solved with the following boundary condition,

$$\varphi^{(0)}(s{=}1) = \pm(2m+1)\frac{\pi}{2} , \tag{45}$$

for any arbitrary integer m. Using the boundary condition (45) the term $d\varphi^{(0)}/ds$ must be small at the plasma wall so that the right-hand side of equation (43) remains finite. Writing $\varphi^{(0)} \pm (2m+1)\pi/2 + \Theta$, we get

$$\cos\Theta\frac{d\Theta}{ds} = -\sin\Theta H(s)\,. \tag{46}$$

Equation (46) can be solved by expanding Θ and $H(s)$ in a Taylor series around $s-1$. In practice one keeps one or two terms in the Taylor expansion because of the complicated dependence of $H(s)$.

4. Conclusion

Local stability analysis of three-dimensional plasmas was presented. The energy principle was considered for strongly elongated modes. The theory presented here is valid below the critical β set by ballooning modes. A function F, proportional to the potential variation, was derived. This function depends on magnetic field key attributes (local magnetic shear, normal and geodesic magnetic curvatures, magnetic field strength and parallel current density) as well as the form of the fluid displacement. For quasi-incompressible motion, the fluid displacement can be found by an expansion method in the low and strong global shear limits.

Our method is particularly suitable for numerical work. A comparative stability analysis of various confinement devices will be reported in a separate paper. Preliminary results indicate that the F-method is typically two orders magnitude faster than full modal calculations (Cooper *et al.* 1996).

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