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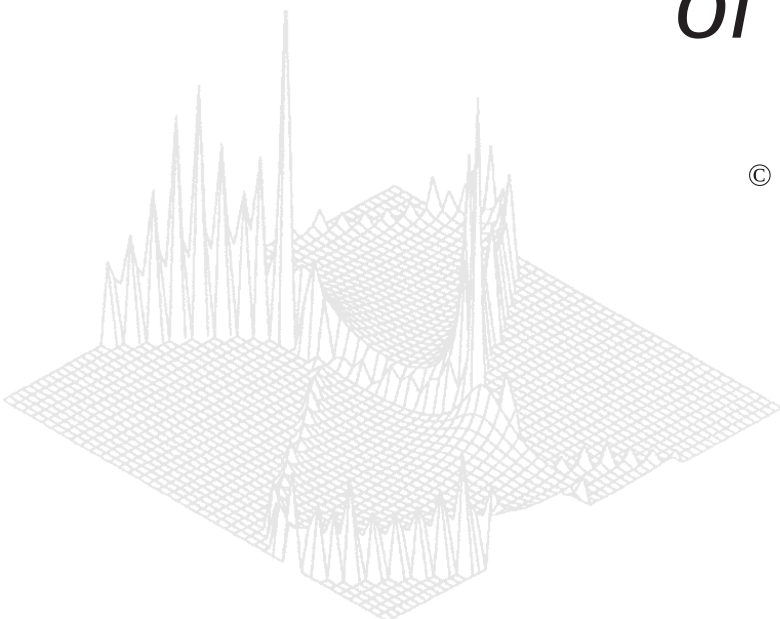

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Some Robertson–Walker Models with Variable G and Λ

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Abstract

Some Robertson–Walker (RW) models admitting a contracted Ricci collineation along the fluid flow vector and having time-varying G and Λ are investigated. The nature of the expansion of the models obtained in the cases $k = \pm 1$ is found to be interchanged from the corresponding standard FRW models. Estimates of the present values of various cosmological parameters are obtained and found to be well within the observational limits.

1. Introduction

The Einstein field equation has two parameters—the gravitational constant G and the cosmological constant Λ. The Newtonian constant of gravitation G plays the role of a coupling constant between geometry and matter in the Einstein field equation. In an evolving universe, it appears natural to look at this 'constant' as a function of time. Numerous suggestions based on different arguments have been proposed in the past few decades in which G varies with time (Wesson 1978, 1980). Dirac (1938, 1975) proposed a theory with variable G motivated by the occurrence of large numbers discovered by Weyl, Eddington and Dirac himself. Many other extensions of Einstein's theory, with time-dependent G, have also been proposed in order to achieve a possible unification of gravitation and elementary particle physics or to incorporate Mach's principle in general relativity (Brans and Dicke 1961; Hoyle and Narlikar 1964; Canuto *et al.* 1977*a*).

From the point of view of incorporating particle physics into Einstein's theory of gravitation, the simplest approach is to interpret the cosmological constant Λ in terms of quantum mechanics and the physics of the vacuum (Zeldovich 1968; Sakharov 1968; Ford 1975; Steeruwitz 1975). The Λ-term arises naturally in general-relativisitic quantum field theory where it is interpreted as the energy density of the vacuum (Zeldovich 1967; Ginzburg *et al.* 1971; Fulling *et al.* 1974). The Λ term has also been interpreted in terms of the Higgs scalar field (Bergmann 1968; Wagoner 1970). Dreitlein (1974) suggested that the mass of the Higgs boson is connected with Λ as well as G. Linde (1974) proposed that Λ is a function of temperature and is related to the process of broken symmetries. It is widely believed that the value of Λ was large during the early stages of evolution and strongly influenced its expansion, whereas its present value is incredibly small (Weinberg 1989; Carroll *et al.* 1992). Several authors, for example Freese *et al.*

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(1987), Ozer and Taha (1987), Gasperini (1987, 1988), Chen and Wu (1990) and Carvalho *et al.* (1992), have advocated a variable Λ in the framework of Einstein's theory to account for this fact. The rationale behind this is that the energy density of the vacuum should spontaneously decay into massive and/or massless particles reducing Λ to its present value. This view is also supported by Gasperini (1987, 1988) who argued that Λ can be interpreted as a measure of the temperature of the cosmic vacuum which should decrease, like the radiation temperature, with cosmic expansion. In this regard, Peebles and Ratra (1988) have shown that if the potential energy of the scalar field during inflation has a power-law tail at large ϕ, the mass density associated with ϕ acts like a cosmological constant that decreases with time.

The possibility of Λ as a function of time has also been considered by Bicknell and Klotz (1976), Canuto *et al.* (1977*b*), Endo and Fukui (1977) and Bertolami (1986) in various variable-G theories in different contexts. Lau (1985) has made Dirac's large number hypothesis compatible with Einstein's theory of gravitation by attempting a tentative generalisation of Einstein field equations with time dependent G and Λ. The possibility of variable G and Λ in Einstein's theory has also been studied by DerSarkissian (1985) by making use of the principle of absolute quark confinement.

Investigating the distance dependence of gravity under very general conditions, Wilkins (1986) found that the gravity field at a distance r from a point mass has two components: one varying as r^{-2}, the other as r (Hookian field). The latter component is identifiable with the weak field limit of the Λ term in Einstein's equations. His analysis allows one to consider both the gravity fields—the Hookian field, coupled to Λ and the Newtonian one, coupled to G—on an equal footing. With this in view, Beesham (1986), Abdel-Rahman (1990) and Kalligas *et al.* (1992) have proposed linking the variation of G with that of Λ in the framework of general relativity. This approach preserves conservation of the energy–momentum tensor of matter and leaves the form of the Einstein field equations unchanged. Although this approach is non-covariant, it is worth studying because it may be a limit of some higher dimensional fully covariant theory (Wesson 1984; Kalligas *et al.* 1992).

In this paper we consider time-dependent G and Λ in the framework of general relativity in the background of RW spacetime and investigate some cosmological models admitting a contracted Ricci collineation along the fluid flow vector by assuming conservation of the energy–momentum tensor of matter content.

2. The Field Equations

We consider the homogeneous and isotropic spacetime described by the RW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + R^2(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right), \tag{1}$$

characterised by its scale factor $R(t)$ and the curvature parameter k $(=\pm 1, 0)$. In the Gaussian normal coordinates of this metric, the fluid flow vector v^i is the normalised timelike eigenvector of T_{ij}, the energy–momentum tensor of the perfect fluid, given by

$$T_{ij} = (\rho + p) v_i \, v_j + p g_{ij} \quad \text{(in units with } c = 1\text{)}. \tag{2}$$

Here ρ is the energy density of cosmic matter and p is its pressure. In the context of (1) and (2), the Einstein field equation, with time-dependent G and Λ, i.e.

$$R_{ij} - \tfrac{1}{2} R^{\ell}_{\ell} \, g_{ij} - \Lambda(t) \, g_{ij} = -8\pi G(t) \, T_{ij} \,, \tag{3}$$

yields two independent equations:

$$-\frac{\ddot{R}}{R} = \frac{4\pi G(t)}{3} (\rho + 3p) - \frac{\Lambda(t)}{3} \,, \tag{4}$$

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G(t)}{3} \rho + \frac{\Lambda}{3} \,, \tag{5}$$

having the same form as in the standard FRW model.

In view of the vanishing divergence of the Einstein tensor, equation (3) gives

$$\dot{\rho} + 3(\rho + p) \frac{\dot{R}}{R} + \rho \frac{\dot{G}}{G} + \frac{\dot{\Lambda}}{8\pi G} = 0 \,. \tag{6}$$

We now assume the law of conservation of energy ($T^{ij}_{;j} = 0$) giving

$$\dot{\rho} + 3(\rho + p) \frac{\dot{R}}{R} = 0 \,, \tag{7}$$

by the use of which, equation (6) yields

$$\dot{G} = -\frac{\dot{\Lambda}}{8\pi\rho} \,, \tag{8}$$

indicating that G increases or decreases according to whether Λ, respectively, decreases or increases. We also consider the perfect fluid equation of state

$$p = w\rho \,, \tag{9}$$

where w, as suggested by Freese *et al.* (1987), may be defined by

$$w = \tfrac{1}{3} \, \frac{\rho_{\mathrm{r}}}{\rho_{\mathrm{m}} + \rho_{\mathrm{r}}} \,, \tag{10}$$

with $\rho = \rho_{\mathrm{m}} + \rho_{\mathrm{r}}$, ρ_{m} and ρ_{r} being the matter (rest mass) and radiation energy densities. As the variation of $w(t)$ is slow compared with the expansion of the universe, except near the time when matter and radiation energy densities are equal, we can approximate $w(t)$ as a step function:

$$w \simeq \begin{cases} \tfrac{1}{3}, & \text{in the radiation dominated (RD) universe} \\ 0, & \text{in the matter dominated (MD) universe} \end{cases} \,. \tag{11}$$

As equations (4)–(9) supply only four independent equations in the five unknowns ρ, p, R, G and Λ, an extra equation is needed to solve the system completely, which we shall obtain in the following by using symmetry considerations.

Symmetry methods of obtaining conservation laws in general relativity are well known. Symmetries of the RW metric belonging to Ricci and contracted Ricci collineations have been discussed by Green *et al.* (1977) and Nunez *et al.* (1990). In this connection, we note that a spacetime is said to admit Ricci collineation along a field vector η^i if $L_\eta R_{ij} = 0$ (Katzin *et al.* 1969), where L_η denotes the Lie derivative along η^i. If the Ricci tensor is Lie transported along η^i, then there exists a conservation law generator of the form (Collinson 1970)

$$[R^j_m \, \eta^m]_{;j} = 0 \,, \tag{12}$$

and if the Einstein field equation (3) is satisfied, this can be written as

$$\left\{ GT^j_m \, \eta^m + \left(\frac{\Lambda}{8\pi} - \tfrac{1}{2} GT \right) \eta^j \right\}_{;j} = 0 \,, \tag{13}$$

which provides the symmetries of the stress energy tensor depending upon the specific character of the symmetry vector η^i.

A spacetime is said to admit a family of contracted Ricci collineation if $g^{ij} L_\eta \ R_{ij} = 0$, which leads to a conservation law generator of the form (Davis *et al.* 1976)

$$\left[\sqrt{-g} \left\{ GT^j_m \, \eta^m + \left(\frac{\Lambda}{8\pi} - \tfrac{1}{2} GT \right) \eta^j \right\} \right]_{;j} = 0 \,. \tag{14}$$

If we specify the symmetry vector η^i by taking it along the fluid flow, i.e. $\eta^i = \text{constant} \times v^i$, then the conservation law generator (14), for T_{ij} given by (2), yields

$$G\left(\rho + 3p - \frac{\Lambda}{4\pi G} \right) R^3 = \text{constant} = A \ (\text{say}) \,, \tag{15}$$

which, in some sense, may be interpreted as the conservation of the total active gravitational mass of the universe (Abdussattar and Vishwakarma 1995, 1996*a*, 1996*b*). From the field equation (4), the conservation law (15) leads to

$$\ddot{R} = -\frac{4\pi A}{3R^2} \,, \tag{16}$$

by virtue of which, $g^{ij} \, \mathrm{L}_\eta \ R_{ij} = 0$ is satisfied identically. On integration, equation (16) gives

$$\dot{R}^2 = \frac{8\pi A}{3R} + B, \qquad B = \text{constant} \,, \tag{17}$$

supplying the time-variation of $R(t)$. Equations (5) and (17) give

$$\rho = \frac{A}{GR^3} + \frac{3(B+k)}{8\pi GR^2} - \frac{\Lambda}{8\pi G} \tag{18}$$

which, taken together with (15), yields

$$p = \frac{\Lambda}{8\pi G} - \frac{B+k}{8\pi GR^2}\,. \tag{19}$$

Abdussattar and Vishwakarma (1996*b*) previously obtained a model of the universe based on these equations, in which the constant A assumes two values: one in the early RD phase and the other in the present MD phase. The model originated from a non-singular origin and was free from the horizon problem. In the present paper, we investigate the prospects of a big bang origin by considering the same A throughout the evolution and obtain three different models for $k = \pm 1, 0$. This will be done in the following section.

3. The Models

By the use of (9), equation (7) may be solved as

$$\rho = CR^{-3(1+w)}; \quad C = \text{Some positive constant}\,, \tag{20}$$

and equations (9), (18), (19) and (20) may be used to obtain

$$G = \frac{R^{3w}}{4\pi(1+w)C}[4\pi A + (B+k)R]\,, \tag{21}$$

$$\Lambda = \frac{1}{(1+w)R^3}[8\pi wA + (1+3w)(B+k)R]\,. \tag{22}$$

In order to determine the constants A, B and C, we associate them with the present values of the cosmological parameters. In this regard we note that the present cosmic energy density $\rho_{\rm p}$, according to current observations (Kolb and Turner 1990; Olive 1990) is close to its corresponding critical value

$$\rho_{\rm cp} \equiv \frac{3}{8\pi G_{\rm p}\, H_{\rm p}^2}\,,$$

$H_{\rm p}$ being the present value of the Hubble parameter. The substitution of the assumption $\rho_{\rm p} = \rho_{\rm cp}$ into equation (5) results in $\Lambda_{\rm p} = 3k/R_{\rm p}^2$. When this result, together with the assumption that $p = 0$ in the MD era, is substituted in equation (19), one obtains

$$B = 2k\,. \tag{23}$$

The value of A then follows from equation (15) as

$$A = G_{\rm p}\,\rho_{\rm p}\,R_{\rm p}^3 - \frac{3k}{4\pi}\,R_{\rm p}\,. \tag{24}$$

The value of C in the MD era may be represented from equation (20) as $C = \rho_{\rm p} R_{\rm p}^3$. The evolution of the deceleration parameter $q \equiv -R\ddot{R}/\dot{R}^2$ is then obtained from equations (16) and (17) as

$$q = \left(2 + \frac{3kR}{2\pi A}\right)^{-1}. \tag{25}$$

Differentiation of equation (21) with respect to t gives

$$\dot{G} = \frac{3R^{3w}\dot{R}}{(1+w)C}\left(\frac{wA}{R} + \frac{k(1+3w)}{4\pi}\right), \tag{26}$$

implying

$$\left[\frac{\dot{G}}{G}\right]_{\rm p} = \frac{3kH_{\rm p}}{4\pi G_{\rm p}\,\rho_{\rm p}\,R_{\rm p}^2}. \tag{27}$$

From the estimates of the parameters as obtained later on, we shall see that $A > 0$ in the models indicating, via equation (16), that $\ddot{R} < 0$. This, with $\dot{R} > 0$, shows that R must have reached $R = 0$ at some finite time in the past, say at $t = 0$, which is a singularity in the present models with $G = 0$ and Λ, ρ and $\dot{R}$ infinite there.

We now consider the three cases $k = 1$, -1 and 0 and investigate some properties of the resulting models.

Case I: k = 1

The time-variation of the scale factor R, in this case, is obtained from equation (17) as

$$R = \frac{2\pi A}{3}(\cosh 2\tau - 1), \qquad t = \frac{\sqrt{2}\pi A}{3}(\sinh 2\tau - 2\tau), \tag{28}$$

which is similar to that in the pressure-less phase of the open ($k = -1$) FRW model in the standard case. Thus $R \to \infty$ as $t \to \infty$, although $k = 1$. Equations (21) and (22) respectively indicate that G increases and Λ decreases, first at a higher rate in the early universe and then comparatively slowly in the present universe. Equation (25) suggests that $q = \frac{1}{2}$ initially. It decreases monotonically and approaches zero as $t \to \infty$. In order to obtain the estimates of the present values of the parameters, we use the observational values $|(\dot{G}/G_{\rm p})| = 10^{-11}\,{\rm yr}^{-1}$, $H_p = 75\,{\rm km\,s^{-1}\,Mpc^{-1}}$ $(=7\cdot5\times10^{-11}\,{\rm yr}^{-1})$, $G_{\rm p} = 6\cdot673\times10^{-8}\,{\rm cm^3\,g^{-1}\,s^{-2}}$ and $\rho_{\rm p} = \rho_{\rm cp} \approx 10^{-29}\,{\rm g\,cm^{-3}}$ in equation (27), giving

$$R_{\rm p} \approx 4\cdot88 \times 10^{28}\,{\rm cm}. \tag{29}$$

This gives $A \approx 7\cdot57\times10^{28}$ cm and hence $q_{\rm p} \approx 0\cdot43$. The present value of the cosmological constant is obtained as $\Lambda_{\rm p} \approx 10^{-57}\,{\rm cm}^{-2}$, which is well within the upper limit of $\Lambda_{\rm p}$ found as $10^{-56}\,{\rm cm}^{-2}$ from the cosmological observations

(Carvalho *et al.* 1992). This is remarkable since the value of Λ is infinitely large initially. The age of the universe is obtained from equation (28) as $t_{\mathrm{p}} \approx 9\cdot 14\times 10^9$ yr which is approximately the same as in the standard model.

Case II: $k = -1$

The metric, in this case, is obtained from equation (17) as

$$R = \frac{2\pi A}{3}\ (1 - \cos 2\tau), \qquad t = \frac{\sqrt{2}\pi A}{3}(2\tau - \sin 2\tau)\,, \tag{30}$$

being similar to that in the pressure-less phase of the closed ($k = 1$) FRW model. Equation (30) indicates that the model attains its maximum radius $R_{\max} = 4\pi A/3$ at $t = \sqrt{2}\pi^2 A/3$ and thereafter starts contracting back to the origin in a way similar to the $k = 1$ case of the standard FRW model, although here $k = -1$. We note from equation (26) that G increases in the early universe, whereas it decreases in the present universe. Thus it reaches a maximum at somewhere in the matter and radiation era where

$$R = \frac{4\pi A}{3}\,\frac{\rho_{\mathrm{r}}}{\rho_{\mathrm{m}} + 2\rho_{\mathrm{r}}}\,.$$

At this time Λ is a minimum as equation (22) indicates. Thereafter G decreases continuously and vanishes at $R = R_{\max}$. It then again starts increasing in the contracting phase. We also note from equation (22) that Λ decreases continuously and vanishes somewhere in the matter and radiation era at

$$R = \frac{8\pi A}{9}\,\frac{\rho_{\mathrm{r}}}{\rho_{\mathrm{m}} + 2\rho_{\mathrm{r}}}\,.$$

Thereafter Λ becomes negative.

It may be noted that the observational values of the parameters as considered in case I, obtain the same R_{p} and hence the same $|\Lambda_{\mathrm{p}}|$ in this model as obtained there. However, this gives $A \approx 9\cdot 90\times 10^{28}$ cm and $q_{\mathrm{p}} \approx 0\cdot 57$ in the present model. We also find that $R_{\max} \approx 4\cdot 15\times 10^{29}$ cm at $t \approx 4\cdot 88\times 10^{11}$ yr. The present age of the universe in this model is obtained from equation (30) as $t_{\mathrm{p}} \approx 8\cdot 66\times 10^9$ yr.

Case III: $k = 0$

In this case R is obtained from equation (17) as

$$R = (6\pi A)^{1/3}\ t^{2/3}\,. \tag{31}$$

This variation of R, which holds throughout the evolution as in the previous cases (28) and (30), is similar to that in the pressure-less phase of the Einstein–de Sitter model. In the present model, G increases as R and Λ decreases as R^{-3} in the early RD era and approach, respectively, G_{p} and 0 as the universe turns matter dominated and the model then reduces to the Einstein–de Sitter model.

4. Conclusions

By taking time-dependent G and Λ in Einstein's theory of general relativity, in a way which conserves the energy–momentum tensor of matter content and leaves the form of the field equations unchanged, we have obtained some models of the universe which admit a contracted Ricci collineation along the fluid flow vector. This imposed symmetry implies the constancy of the so-called active gravitational mass of the universe. In earlier work (Abdussattar and Vishwakarma 1996*b*), we have taken this constant changing with phases. Here we have assumed the same constant throughout the evolution which obviously results in obtaining the same model describing the different phases of evolution.

Another consequence of the assumed symmetry is that the curvature parameter k, for $k = \pm l$, has an opposite role, as compared to the standard model, in determining the nature of expansion. For $k = 0$, however, the model reduces to the standard (Einstein–de Sitter) model in the present matter dominated phase, though its early evolution is altogether different.

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