

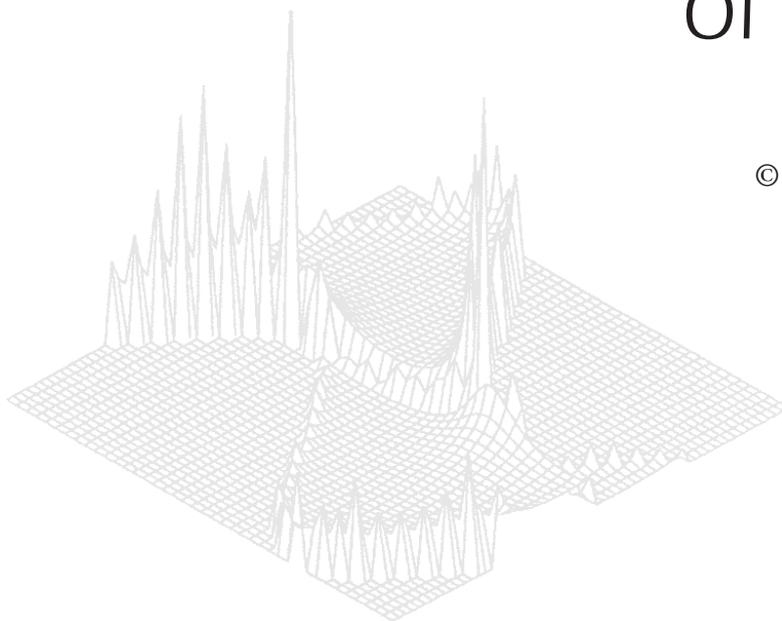
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# Nonequilibrium Process in the $\sigma$ Model and Chemical Relaxation Time in a Homogeneous Pionic Gas\*

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## *Abstract*

In a homogeneous pionic gas system, a chemical nonequilibrium process is understood to have an effect in the expansion processes that are realized immediately after heavy ion collisions. The chemical relaxation time is calculated by incorporating the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction, which is given in the second order of perturbation in the  $\sigma$  model. The  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction is assumed to be less frequent than the  $\pi + \pi \leftrightarrow \pi + \pi$  scattering that is expected to establish the local equilibrium, and the hydrodynamical equation is solved for various initial conditions. It is shown that the relaxation time is of the order of 100 fm and does not have a significant effect on the expansion process, which implies that the pion number freezeout takes place at an early stage of the expansion.

## 1. Introduction

It is widely believed that quantum chromodynamics (QCD) is the underlying theory of the strong interaction and that QCD exhibits a confinement–deconfinement phase transition at high temperature ( $\approx 200$  MeV) [1]. Since understanding the properties of this phase transition is crucial for hadron physics, many efforts have been made both experimentally and theoretically [2]. In particular, a heavy ion reaction is expected to be a useful method to establish high temperatures with free quarks and gluons and to investigate the process of hadronization.

Heavy ion collisions are considered to undergo several steps:

- (1) Two highly Lorentz contracted nuclei collide with each other.
- (2) They pass through each other and a hot system is formed in between, where quarks and gluons are excited [3].
- (3) Frequent collisions and reactions among the excited elements lead to local equilibrium. Therefore this state can be treated by hydrodynamics [4, 5].
- (4) As the system expands, cooling and a subsequent phase transition process proceeds [2].
- (5) A lot of pions (and other light hadrons) are created under the process of the phase transition [6].
- (6) When the phase transition terminates, there exists a hadronic gas which can also be treated by hydrodynamics. Since the system is hot, the

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collisions and reactions between the hadrons are frequent and the total number of hadrons is not conserved.

- (7) As the cooling proceeds, reactions which change the total number of hadrons become unlikely. At this stage, the multiplicity of pion (and other light hadrons) is fixed (freezeout of number) [7].
- (8) When the expansion proceeds further and the mean distance between pions is equivalent to the average variation of macroscopic quantity such as temperature, energy density and pressure, the hydrodynamical picture is not a good one any longer. Instead, the system can be considered to be composed of free pions without any interaction. This stage is called the freezeout of temperature.

Therefore, the multiplicity of pions is determined at stage 7, not at the time when the phase transition terminates. Also, the energy distribution of pions is given by the thermal distribution at stage 8. In other words, the hadronic final state of heavy ion collisions does not directly reflect the property of the confinement–deconfinement phase transition.

Due to this feature, it has been proposed to use an electromagnetic and weak probe, which is free from the hadronic final state interaction, to investigate the hadronic property of the phase transition. However, if we can understand the process of the final state, we can extract the information on the phase transition by hadronic probes. In particular, pion multiplicity can be used to obtain the pion number and the entropy immediately after the phase transition: if one knows the elementary chemical process to change the pion multiplicity such as  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$ , one can see how the pion multiplicity varies through the cooling process. From the information on the pion multiplicity obtained in heavy ion reaction one can then track the process backward to the time when the phase transition is over.

Whether the chemical reaction largely affects the cooling process or not depends on the balance between the cooling by expansion and the speed of the chemical reaction. If the typical time scale of the chemical reaction is longer than that of expansion, the freezeout of pion multiplicity takes place at an early stage of the expansion and one can identify the final pion multiplicity with the pion multiplicity immediately after the phase transition in the first approximation. In this case we can estimate the entropy density immediately after the phase transition  $s$  by

$$s = 4n, \quad n : \text{observed pion multiplicity}, \quad (1)$$

which is obtained for massless classical particle systems with the particle numbers conserved [8]. If the former is shorter than the latter, the chemical reaction remains dominant through the expansion. In this case, the final multiplicity is quite different from the initial one at the end of the phase transition, and Eq. (1) cannot be used to obtain the entropy density. Instead, one must track the irreversible process backward, estimate the pion multiplicity at the end of the phase transition and then use Eq. (1).

In this paper, we aim to obtain the speed of the chemical reaction  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  based on the  $\sigma$  model. In this model, the interaction in the leading order of perturbation just gives  $\pi + \pi \leftrightarrow \pi + \pi$  scattering, which is

expected to establish local equilibrium in a pionic gas. In the second order of perturbation, the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction is derived. If the latter reaction is less frequent than the former, we can resort to the hydrodynamical equation to track the change of pion number multiplicity. In Section 2 an evaluation of the frequency of these two reactions is made and the criterion to establish local equilibrium is given. Then, we present the hydrodynamical equation in the classical approximation. In Section 3 the hydrodynamical equation is solved numerically in a homogeneous pion gas for various initial temperatures and chemical potentials. The typical time scale of the chemical reaction (relaxation time) is obtained and compared to the expansion rate. The last section is devoted to a summary and discussing open problems.

## 2. Elementary Chemical Reaction in the $\sigma$ Model and the Hydrodynamical Equation

We take the  $\sigma$  model to describe the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction. The interaction part of the Lagrangian is given by

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4!}(\sigma^2 + \pi^2 - f_\pi^2)^2, \quad (2)$$

where  $\pi = \pi^1\tau_1 + \pi^2\tau_2 + \pi^3\tau_3$  and  $f_\pi$  is the pion decay constant. After shifting the  $\sigma$  field as  $\sigma \rightarrow \sigma + f_\pi$ , one obtains  $-\lambda/4!(\pi^2 + 2f_\pi\sigma + \sigma^2)^2$  as the potential term. This term gives  $\pi + \pi \leftrightarrow \pi + \pi$  scattering as depicted in Fig. 1a. Summing all four diagrams in Fig. 1a at threshold  $p_i = (m, 0, 0, 0)$  ( $i = 1, 2, 3, 4$ ) gives

$$-\lambda + \left(\frac{\lambda}{3}\right)^2 \frac{f_\pi^2}{m_\sigma^2} + \left(\frac{\lambda}{3}\right)^2 \frac{f_\pi^2}{m_\sigma^2} + \left(\frac{\lambda}{3}\right)^2 \frac{f_\pi^2}{m_\sigma^2 - (2m)^2}, \quad (3)$$

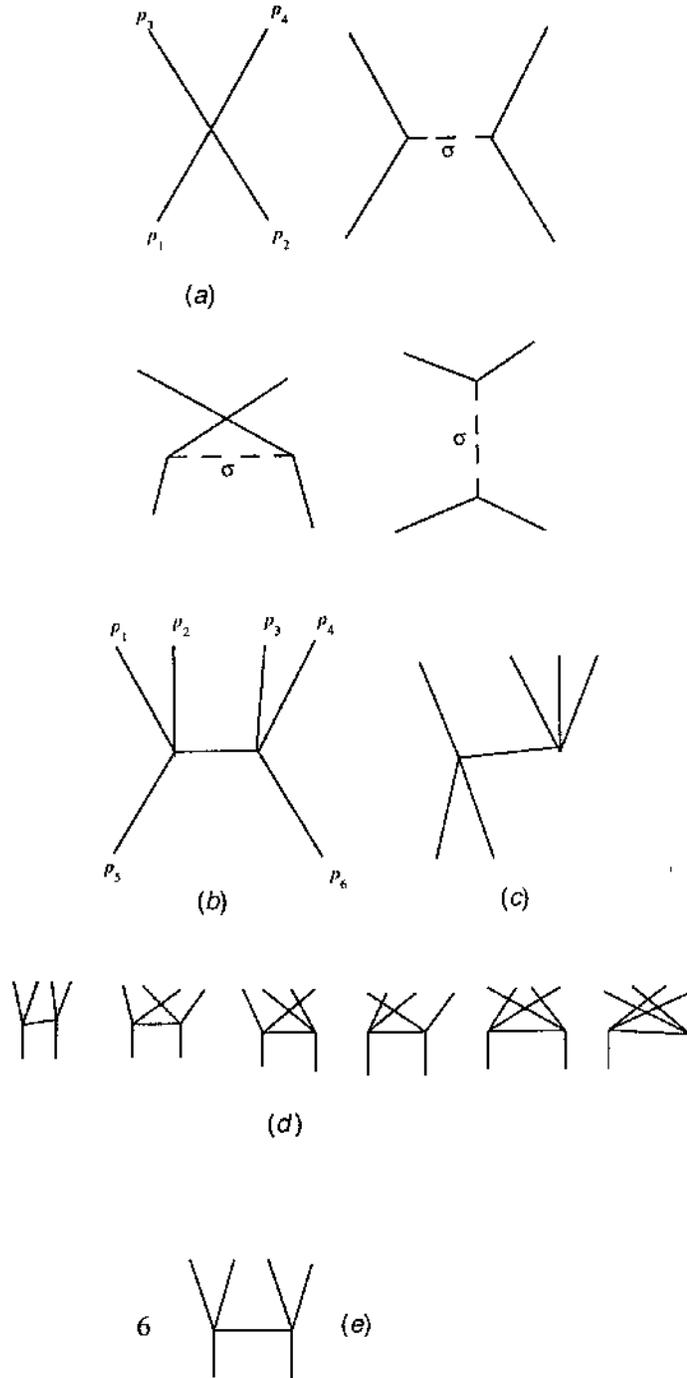
where  $m_\sigma$  ( $m$ ) is the  $\sigma$  ( $\pi$ ) meson mass. Since  $m_\sigma^2 = \lambda f_\pi^2/3$ , we get

$$-\frac{\lambda}{3} + \frac{\lambda}{3} \frac{1}{1 - (2m/m_\sigma)^2} \quad (4)$$

as an effective  $4\pi$  coupling constant. For  $m_\sigma \approx 600$  MeV,  $m/m_\sigma \approx \frac{1}{4}$  and the effective coupling constant is  $\lambda/9$ . For simplicity, we hereafter replace the sum of Fig. 1a by just one direct  $4\pi$  coupling with its coupling constant  $\lambda/9$ .

The  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction is given in the second order of perturbation (Fig. 1b). In second order, another diagram is possible for the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  process (Fig. 1c). This diagram, however, is not taken into account because the denominator of the propagator  $p^2 - m^2$  is of the order of  $(3m)^2 - m^2$  and the contribution of Fig. 1c is  $O(1/10)$  of Fig. 1b. All the possible diagrams of type Fig. 1b are shown in Fig. 1d. For simplicity, we replace Fig. 1d by Fig. 1e, which means the effect of interference is not properly taken into account.

In the next section, the hydrodynamical equation is solved assuming a local equilibrium that is expected to be established by the  $\pi + \pi \leftrightarrow \pi + \pi$  scattering. Since  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  drives the system away from equilibrium,  $\pi + \pi \leftrightarrow \pi + \pi$  scattering must be frequent enough so that the disturbed nonequilibrium state can swiftly return to the equilibrium state.



**Fig. 1.** Elementary processes under consideration: (a)  $\pi + \pi \leftrightarrow \pi + \pi$  scattering. (b)  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  process. (c) The other possible  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  process in the second order of perturbation in the  $\sigma$  model. This diagram is neglected. (d) All the possible detailed diagrams of type (b). (e) Approximated diagram for (d) used in the numerical calculation. All the 4 $\pi$  vertices in (b), (c), (d), (e) are the effective coupling obtained from the sum of diagrams in (a).

Once local equilibrium is established, the local temperature  $T(x)$  as well as the local chemical potential  $\mu(x)$  is defined. The  $\pi + \pi \leftrightarrow \pi + \pi$  actually means  $\pi^0 + \pi^0 \leftrightarrow \pi^+ + \pi^-$  and  $\pi^i + \pi^j \leftrightarrow \pi^i + \pi^j$  ( $i, j = +, -, 0$ ). The first one gives the equilibrium condition  $2\mu^0(x) = \mu^+(x) + \mu^-(x)$ . Since charge conservation means  $\mu^+(x) = \mu^-(x)$  in the neutral pionic gas, we get  $\mu^+(x) = \mu^-(x) = \mu^0(x)$  or  $n^{\pi^+}(x) = n^{\pi^-}(x) = n^{\pi^0}(x)$ . This means that all the equations are identical for  $\pi^0, \pi^+$  and  $\pi^-$ .

In other words, all the calculation is carried out in the same way as in  $\lambda\phi^4$  theory, except for a numerical factor stemming from the isospin degrees of freedom, because the essential part of the  $\sigma$  model is only the  $(\lambda/4!)\pi^4$  term.

As we have mentioned before, the  $\pi + \pi \leftrightarrow \pi + \pi$  scattering should be more frequent than the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction. The frequency of the  $\pi + \pi \leftrightarrow \pi + \pi$  (Fig. 1a) and  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  (Fig. 1e) process, denoted by  $I^{\pi+\pi \leftrightarrow \pi+\pi}$  and  $I^{\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi}$  and defined by

$$I^{\pi+\pi \leftrightarrow \pi+\pi} \equiv \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} \frac{d\mathbf{p}_4}{(2\pi)^3} \left(\frac{\lambda}{9}\right)^2 (2\pi)^4 \delta(p_1 + p_2 - p_3 - p_4), \quad (5)$$

$$\begin{aligned} I^{\pi+\pi \leftrightarrow \pi+\pi+\pi+\pi} &\equiv \int \frac{d\mathbf{p}_1}{(2\pi)^3} \frac{d\mathbf{p}_2}{(2\pi)^3} \frac{d\mathbf{p}_3}{(2\pi)^3} \frac{d\mathbf{p}_4}{(2\pi)^3} \frac{d\mathbf{p}_5}{(2\pi)^3} \frac{d\mathbf{p}_6}{(2\pi)^3} \\ &\times |6M|^2 (f(p_5)f(p_6) - f(p_1)f(p_2)f(p_3)f(p_4)) \\ &\times (2\pi)^4 \delta(p_1 + p_2 + p_3 + p_4 - p_5 - p_6), \end{aligned} \quad (6)$$

$$f(p) = \exp\{-\beta(\sqrt{\mathbf{p}^2 + m^2} - \mu)\}$$

$$\beta = \frac{1}{T}, \quad T : \text{temperature}, \quad \mu : \text{chemical potential}$$

$M$  : the matrix element of the diagram in Fig. 1b

$$= \left(\frac{\lambda}{9}\right)^2 \frac{1}{m^2 - (p_1 + p_2 - p_5)^2}, \quad (7)$$

are calculated essentially numerically. To get a rough estimate for  $I^{\pi+\pi \leftrightarrow \pi+\pi}$  and  $I$ , we took  $m\beta \approx 1$  and  $\mu/m \approx 1$  and found that  $\lambda \ll 1000$  is required for  $I^{\pi+\pi \leftrightarrow \pi+\pi} \gg I$ . Since  $\lambda \approx 100$  is taken for the ordinary  $\sigma$ -model, the local equilibrium is established in the present case.\*

Under local equilibrium, relativistic hydrodynamical equation [9] is given as

\* Since  $\pi + \pi \leftrightarrow \pi + \pi$  is  $O(\lambda^2)$  and  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  is  $O(\lambda^4)$ , it may seem that  $\lambda \ll 1$  is required for local equilibrium. However, numerical factors coming from the phase space integral such as  $1/(2\pi)^3$  for each external line and  $e^{-4\beta}$  in Eq. (6) suppress the frequency of the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  process. Thus  $\lambda \approx 100$  satisfies the criterion for local equilibrium even if  $\lambda$  is bigger than 1.

$$\partial_\mu N^{i\mu} = 2I, \quad (8)$$

$$\partial_\mu T^{i\mu\nu} = 0, \quad (9)$$

for each species of pion, where  $N^{i\mu}$  ( $i = +, -, 0$ ) is the pion number current and  $T^{i\mu\nu}$  is the energy-momentum tensor for  $\pi^+$ ,  $\pi^-$  and  $\pi^0$  respectively. As mentioned above, these fundamental equations take the same form for all species of pions. Hence, we hereafter suppress the indices  $i = +, -, 0$ . Eq. (8) describes the change of the total pion number and is called the rate equation, whose right-hand side  $2I$  gives the rate of particle production and annihilation. The factor of 2 comes from the fact that two pions are created for one elementary reaction in Fig. 1b. Eq. (9) is the energy-momentum conservation identity.

The  $N^\mu$  and  $T^{\mu\nu}$  are expressed in terms of the pion distribution function  $f$ :

$$N^\mu = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^\mu}{p^0} f, \quad (10)$$

$$T^{\mu\nu} = \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^\mu p^\nu}{p^0} f. \quad (11)$$

### 3. Numerical Solution for the Hydrodynamical Equation and Chemical Relaxation Time in a Homogeneous Pionic Gas

In this section we consider homogeneous pionic gases to obtain the typical time scale of the chemical reaction  $\pi + \pi \leftrightarrow \pi + \pi$ . In a homogeneous gas, the hydrodynamical equations Eqs. (8) and (9) are drastically simplified to [4]

$$\frac{dn}{dt} = 2I, \quad (12)$$

$$\frac{d\epsilon}{dt} = 0, \quad (13)$$

where  $n = N^0$  ( $\epsilon = T^{00}$ ) is the pion density (energy density). To express  $N^\mu$  and  $T^{\mu\nu}$  in terms of the local temperature  $T(x)$  and local chemical potential  $\mu(x)$ , we need to have the explicit form of the distribution function  $f$ . In this paper we take classical distribution

$$f(p, x, t) = \exp\{-\beta(x, t)(\sqrt{\mathbf{p}^2 + m^2} - \mu(x, t))\}. \quad (14)$$

Since we investigate a homogeneous pionic gas, there is no  $x$  dependence in any macroscopic quantity, and  $T$  and  $\mu$  are functions of time  $t$  alone. Then, we get

$$n(t) = \frac{1}{2\pi^2} e^{\mu/T} T m^2 K_2(m/T), \quad (15)$$

$$\epsilon(t) = \frac{1}{2\pi^2} e^{\mu/T} m^2 \{3T^2 K_2(m/T) + T m K_1(m/T)\}. \quad (16)$$

Combining Eqs (12) and (13) with Eqs (15) and (16), one gets a set of differential equations for  $T(t)$  and  $\mu(t)$ . The energy conservation law Eq. (13) means  $\epsilon(t) = \epsilon(t=0)$  giving the equation to determine  $\mu(t)$  from  $T(t)$  and initial conditions  $T_0 = T(0), \mu_0 = \mu(0)$ :

$$\mu = \frac{\mu_0}{T_0} T + T \ln \left\{ \left( \frac{T_0}{T} \right)^2 \frac{3K_2(m/T_0) + m/T_0 K_1(m/T_0)}{3K_2(m/T) + m/T K_1(m/T)} \right\}, \quad (17)$$

with the abbreviation  $\mu = \mu(t), T = T(t)$ . Also from  $d\epsilon/dt = 0$ , we can express  $d\mu/dt$  in terms of  $T(t), \mu(t), dT/dt, T_0$  and  $\mu_0$ :

$$\dot{\mu} = \left( \frac{\mu}{T} - A \right) \dot{T}, \quad (18)$$

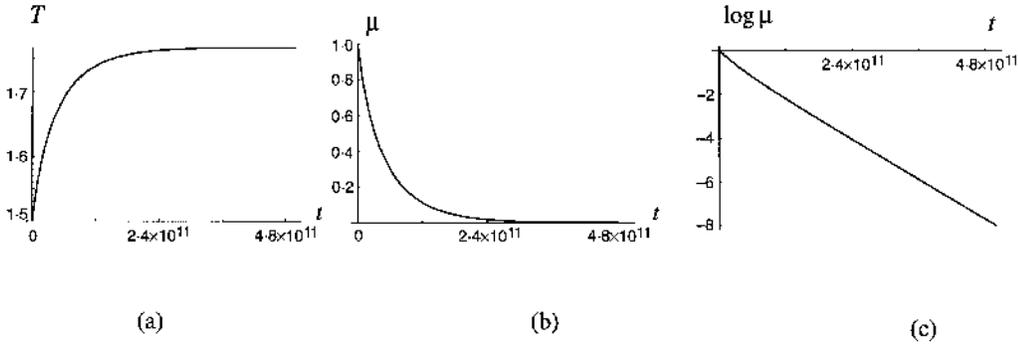
$$A \equiv \frac{12K_2(m/T) + 5m/T K_1(m/T) + (m^2/T^2)K_0(m/T)}{3K_2(m/T) + (m/T)K_1(m/T)}, \quad (19)$$

where  $\dot{\mu}$  ( $\dot{T}$ ) means  $d\mu/dt$  ( $dT/dt$ ). By eliminating  $\mu$  and  $\dot{\mu}$  and redefining all the quantities to be dimensionless, i.e.  $T/m \rightarrow T \equiv 1/\beta, \mu/m \rightarrow \mu, \lambda^4 m t \rightarrow t$ , we can rewrite the hydrodynamical equation as

$$\{(3 - A)K_2(\beta) + \beta K_1(\beta)\} \dot{T} = 2I/\lambda^4. \quad (20)$$

Since  $I$  is proportional to  $\lambda^4$  [see Eq. (6)], the  $\lambda^4$  term in the redefinition of  $t$  makes the equation independent of  $\lambda$ .

By solving this differential equation, one gets the  $t$  dependence of  $T$  and from Eq. (17)  $\mu(t)$  is also obtained. A typical result of numerical calculation is shown in Fig. 2 for initial conditions  $(T_0, \mu_0) = (1.5, 1)$ .



**Fig. 2.** (a) Time dependence of temperature  $T$ , (b) chemical potential  $\mu$  and (c)  $\log|\mu|$  for initial conditions  $(T_0, \mu_0) = (1.5, 1.0)$ . The temperature  $T$ , chemical potential  $\mu$ , and time  $t$  are all dimensionless as defined in the text.

For other initial conditions, relaxation times defined by  $\mu(t_{\text{relax}}) = \mu_0/e$  with  $\lambda = 120$  are shown in Fig. 3. Here  $t_{\text{relax}}$  is shorter for denser and hotter the systems and the shortest one is of the order of 100 fm. This time scale is

expected to be larger than the typical time scale of expansion. Thus, we have found that the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction does not play an important role in an expanding pionic gas system. In other words, the pion number freezes out at an early stage of the final state of a heavy ion reaction. This implies that the hadronic probe is a good one to extract information on the phase transition and that  $s = 4n$  is an appropriate one to evaluate the entropy density immediately after the phase transition.

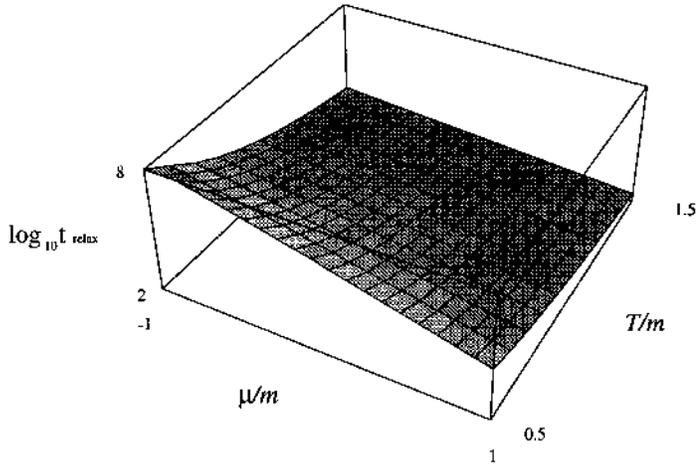


Fig. 3. Plot of  $\log_{10} t_{\text{relax}}$  for various initial conditions with  $\lambda = 120$ .

#### 4. Summary and Open Problems

We have investigated the effect of the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  chemical reaction in the framework of hydrodynamics. To understand the typical time scale of this reaction, we calculated the relaxation times in homogeneous pionic gas systems. It turns out that the relaxation time is of the order of 100 fm at shortest and is longer than the typical time scale of the expansion of the pionic gas.

Although we found that the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  chemical reaction does not play an important role within the framework of the  $\sigma$ -model, there are still numerous problems to solve. The first one is to take into account the effect of the baryon in the central region, which has been neglected so far. In the real case, thermally excited  $N\bar{N}$  pairs exist in the central region. Since the  $\pi$ - $N$  interaction is much larger than the  $\pi\pi$  interaction, the processes such as  $N + \pi \leftrightarrow N + \pi + \pi$  might be significant even if the number of the excited  $N\bar{N}$  pairs is small. Furthermore, if there is a stopping in heavy ion collisions and there exists baryon number coming from the colliding nuclei, there would be a big change of pion multiplicity.

Secondly, to make a reliable quantitative discussion on the role of the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction, the calculation must be made without classical approximation incorporating the effect of interference. This is expected to make the collision less frequent and lead to a larger relaxation time, keeping the essential statement in the former section. The third problem is to take the medium effect

into account. So far, we have fixed the parameters in the  $\sigma$ -model: the coupling constant  $\lambda$  and the pion mass  $m$ . It is known, however, that these quantities depend on the temperature and chemical potential. The correction due to this effect might not be negligible.

Even if some of these effects lead to a smaller relaxation time and make the effect of the change of the pion number a little more important, we still do not expect them to change our results qualitatively for the following reason. In an expanding system, the chemical potential at the end of the phase transition is taken to be 0, because there is a strong interaction between a quark-gluon plasma and pions and the pion can change its multiplicity easily. Therefore, at an early stage of the expansion in which the system is hot enough, the chemical potential is close to 0 and the system is close to chemical equilibrium. So, the  $\pi + \pi \leftrightarrow \pi + \pi + \pi + \pi$  reaction cannot play an important role. As the system expands, the chemical potential deviates from 0. However, the system is not hot enough and the relaxation time is too long to make the effect of reaction significant.

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