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Direct current electrical potentials in an arbitrarily anisotropic medium

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ABSTRACT

In this paper, electrical potential due to a point source in an arbitrarily anisotropic medium is obtained by means of converting the relevant matrix equation into a Poisson's equation through an orthogonal coordinate transformation and a stretching transformation. In a geophysical exploration, depending upon the depth of a mineral deposit (target) from the plane of electrical potential measurements, the effects of electrical anisotropy may result in missing a target.

INTRODUCTION

A material is electrically anisotropic if its electrical characteristics are direction dependent. Different mechanisms producing preferred alignments of crystals or minerals in the rocks, and differentially aligned cracks (or pores under pressure) may give rise to anisotropy in the properties of the materials. As these mechanisms were invariably present in the formation of the earth's crust, anisotropy in lithospheric materials is a widespread phenomenon. Electrical and electromagnetic field measurements by geophysical methods are considerably influenced by the anisotropy in the electrical properties of the layers in the crust. In isotropic materials, the electric field E_i and the total volume density of electric current J_i, $\{i=1,2,3\}$ are in the same direction and their relationship is expressed by

$$J_i = \sigma E_i, \tag{1}$$

where σ the conductivity of the medium, is a scalar quantity. This relation shows that by measuring the electric field the effect of the medium on the measurements can be easily recognised. In an anisotropic medium, directional dependent field quantities are related through the following constitutive relation:

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} \\ \sigma_{2,1} & \sigma_{2,2} & \sigma_{2,3} \\ \sigma_{3,1} & \sigma_{3,2} & \sigma_{3,3} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \tag{2}$$

where the numbers 1, 2 and 3 denote the three fixed orthogonal directions of a Cartesian coordinate system. Here $\{\sigma_{1,1}, \sigma_{1,2}, \ldots \sigma_{3,3}\}$are the elements of the conductivity tensor which is symmetric and positive definite. From equation (2) it is seen that electric field measurements depend on the direction of measurements in an electrically anisotropic material. If we ignore anisotropic effect of the material and interpret the geophysical measurements with the assumptions that the layers in the subsurface are isotropic, our interpretations could go wrong (missing subsurface targets in drilling). For better understanding of the field measurements, effects of anisotropy should be considered in the interpretation of the field data.

To study the anisotropic effects, a higher order symmetry in the conductivity tensor has been considered by several workers by assuming that the medium is *transversely isotropic*, i.e., the principal axes of anisotropy are parallel to and perpendicular to the earth's surface (Bhattacharya and Patra, 1968; Asten, 1974; Srinivas and Upadhayaya, 1974; Bhattacharya and Sen, 1981; Eloranta, 1988; Negi and Saraf, 1989). Recently, considering azimuthal anisotropy, direct current electrical potentials of an anisotropic half-space have been computed by Lindell et al. (1993). Computation of electrical potential in a medium in which only the diagonal elements of the conductivity tensor exist, is given in Parasnis (1986, Appendix 10, p.374). Although these are useful considerations in studying anisotropy effects in electrical measurements, they are too restrictive for many materials in the crust of the earth. For example, the surface layers in tropical and subtropical countries are deeply weathered and display a more complicated anisotropy because of developing cracks. Such cases may be common in countries like Australia where the mineral deposits lie under a weathered conductive surface layer. It is important to consider the presence of strongly arbitrarily anisotropic media and their influences in the interpretation of electrical and electromagnetic measurements by geophysical methods. In this note an analytical solution is presented for studying the effect of arbitrarily (oblique) anisotropic conducting material in electrical potential measurements as a result of direct current distribution in the medium.

MATHEMATICAL FORMULATION

Let us consider a homogeneous, arbitrarily anisotropic, medium into which a point source injects an amount of electric current I. To specify position in the anisotropic medium, the coordinates $\{x_1, x_2, x_3\}$ with respect to a fixed, orthogonal, Cartesian reference frame with the origin O and the three mutually perpendicular base vectors $\{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$ of unit length each, will be considered; $\mathbf{i}_3$ points vertically downward. The subscript notation for vectors and tensors is used and the summation convention applies. Lowercase Latin subscripts are to be assigned the values 1, 2, 3, while a Greek subscript is to be assigned the values 1, 2. Partial differentiation is denoted by ∂; ∂_m denotes differentiation with respect to x_m.

For the static electric current flow, the field quantities are governed by the following equations:

$$\partial_{x_r} V(\boldsymbol{x}) = -E_{x_r}(\boldsymbol{x}), \tag{3}$$

$$\partial_{x_k} J_{x_k}(\boldsymbol{x}) = I\delta(\boldsymbol{x}-\boldsymbol{x}^s), \tag{4}$$

where, V is the electric potential, $\boldsymbol{x}$ and $\boldsymbol{x}^s$ denote the coordinates of the field point and the point source and $\delta(\boldsymbol{x}-\boldsymbol{x}^s)$ is Kronecker symbol, $\delta(\boldsymbol{x}-\boldsymbol{x}^s)=1$ for $(\boldsymbol{x}=\boldsymbol{x}^s)$. Now, we write the constitutive relation, equation (2), in tensorial notation as

$$J_k = \sigma_{k,r} E_r, \tag{5}$$

which can also be written as

$$E_r = \zeta_{r,k} J_k \tag{6}$$

with the relationship,

$$\zeta_{r,k} = (\sigma^{-1})_{k,r}. \tag{7}$$

Here, $\zeta_{r,k}$ is the resistivity tensor which is the inverse of the conductivity tensor $\sigma_{k,r}$ and both of them are symmetric and positive definite. From equations (3), (4), and (5), we write

$$\sigma_{k,r}\partial_{x_k}\partial_{x_r} V = -I\delta(\boldsymbol{x}-\boldsymbol{x}^s). \tag{8}$$

Here onwards, we drop the spatial argument of V. Equation (8) is, in fact, Poisson's equation for the medium under consideration. Using the following orthogonal coordinate transformation

$$y_n = \beta_{n,m}(x_m - x_m^s) \tag{9}$$

the left-hand side operation on V of equation (8) is changed as in the following:

$$\sigma_{k,r}\partial_{x_k}\partial_{x_r} = \gamma^{(n)}\delta_{n,s}\partial_{y_n}\partial_{y_s}, \tag{10}$$

where,

$$\beta_{n,k}\sigma_{k,r}\beta_{s,r} = \gamma^{(n)}\delta_{n,s}. \tag{11}$$

Equation (11) reduces to

$$\sigma_{p,r}\beta_{s,r} = \gamma^{(s)}\beta_{s,p}. \tag{12}$$

This relation defines an eigenvalue problem for the matrix $\sigma_{p,r}$ with eigenvalues $\gamma^{(s)}$;s=1,2,3 and the corresponding eigen-column vector $\{\beta_{s,r};s=1,2,3\}$ In view of the positive definiteness of $\sigma_{k,r}$, all $\gamma^{(s)}$ are real and positive. Further, since $(\beta_{n,m})$ is orthogonal, we have $\det(\beta_{n,m})=1$ and hence

$$\delta(\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{x}^s). \tag{13}$$

Collecting the results, we arrive at

$$\gamma^{(k)}\partial_{y_k}\partial_{y_k} V = -I\delta(\boldsymbol{y}). \tag{14}$$

Next, we apply the stretching transformation

$$\frac{1}{\sqrt{\gamma^{(k)}}} y_k = z_k, \tag{15}$$

which yields

$$\gamma^{(k)}\partial_{y_k}\partial_{y_k} = \frac{\partial}{\partial_{z_k}}\frac{\partial}{\partial_{z_k}}, \tag{16}$$

and

$$\delta(\boldsymbol{y}) = \frac{1}{\sqrt{\gamma^{(1)}\gamma^{(3)}\gamma^{(3)}}}\delta(\boldsymbol{z}). \tag{17}$$

Equations (14), (16), and (17) yield

$$\partial_{z_k}\partial_{z_k} V = -\frac{I}{\sqrt{\gamma^{(1)}\gamma^{(3)}\gamma^{(3)}}}\delta(\boldsymbol{z}). \tag{18}$$

Now, equations (18) is nothing but Poisson's equation for a point source with strength $\frac{I}{\sqrt{\gamma^{(1)}\gamma^{(3)}\gamma^{(3)}}}$, the solution of which is

$$V = \frac{I}{\sqrt{\gamma^{(1)}\gamma^{(3)}\gamma^{(3)}}}\frac{1}{4\pi|\boldsymbol{z}|}. \tag{19}$$

From equations (9) and (15)

$$|\boldsymbol{z}| = [\beta_{k,i}(\gamma^{(k)})^{-1}\beta_{k,j}(x_i - x_i^s)(x_j - x_j^s)]^{\frac{1}{2}}. \tag{20}$$

Using equations (7), (12), and (20)

$$|\boldsymbol{z}| = [(x_i - x_i^s)\zeta_{i,j}(x_j - x_j^s)]^{\frac{1}{2}}. \tag{21}$$

Further, $\gamma^{(s)}$;s being the eigenvalues of $\sigma_{p,r}$ (inverse of the matrix $\zeta_{p,r}$), we have

$$\frac{1}{\sqrt{\gamma^{(1)}\gamma^{(3)}\gamma^{(3)}}} = (\det[\zeta_{p,r}])^{\frac{1}{2}}. \tag{22}$$

Finally, from equations (19), (21), and (22)

$$V = \frac{1}{4\pi}\frac{(\det[\zeta_{p,r}])^{\frac{1}{2}}}{[(x_i - x_i^s)\zeta_{i,j}(x_j - x_j^s)]^{\frac{1}{2}}}. \tag{23}$$

Form equations (23) we observe that

$$(x_i - x_i^s)\zeta_{i,j}(x_j - x_j^s) = \text{constant}, \tag{24}$$

which means that the surfaces of constant potentials are the ellipsoids as $x_r\zeta_{r,k}x_k$ is constant. These ellipsoids will be centred at x_i=x_{is}. Their axial directions are the eigenvectors of $\zeta_{p,q}$. If we cut the set of equipotential surfaces by a plane, concentric ellipses of equipotential curves will lie on the plane.

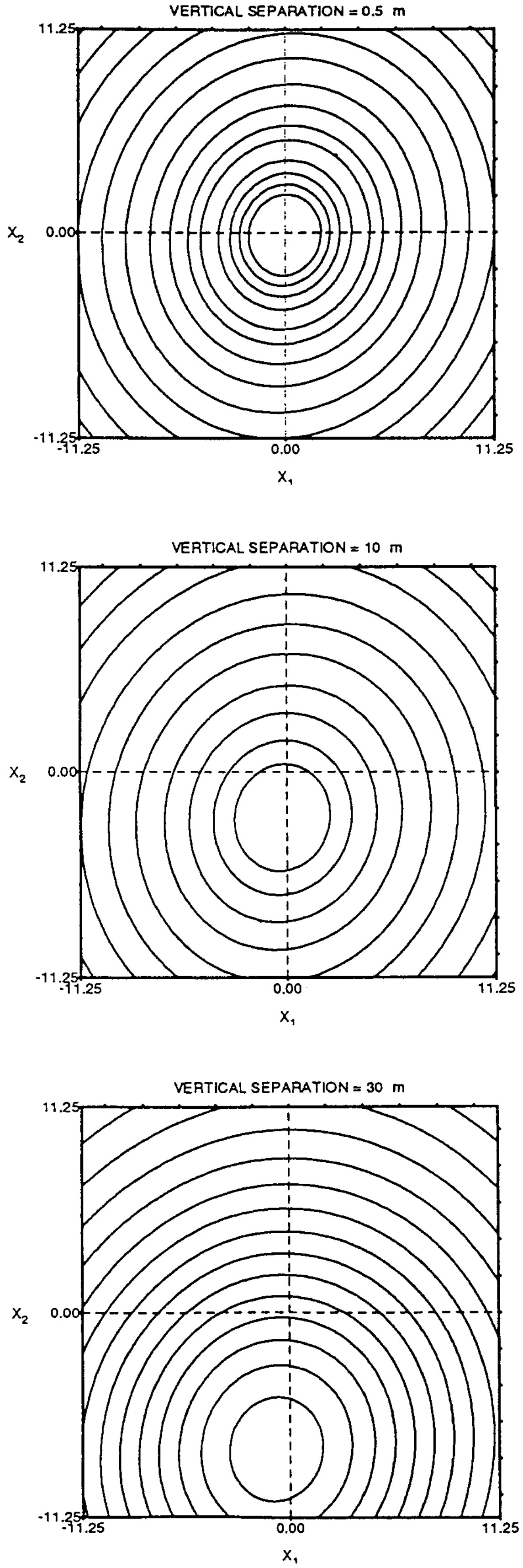

Fig. 1. Equipotential contours of electrical potentials on a horizontal plane in the arbitrarily anisotropic medium; the depths of the point source are 0.5, 10 and 20 m, vertically below the centre of the horizontal plane.

NUMERICAL COMPUTATIONS

We consider an arbitrarily anisotropic medium which has the following electrical conductivity parameters:

$$[\sigma] = \begin{pmatrix} 0.1 & 0.005 & 0.1 \\ 0.005 & 0.08 & 0.01 \\ 0.1 & 0.01 & 0.5 \end{pmatrix} \mathrm{S/m}. \tag{25}$$

Let an electric point source inject direct current in this obliquely anisotropic medium. In Figure 1, the equipotential contours on the horizontal planes at different vertical separations, i.e., 0.5 m, 10 m, and 30 m from the point source are presented. The point source is located vertically below the crossing of the two broken lines on each horizontal plane of measurements. It is observed that with the increasing vertical separation between the point source and the plane of measurements, the centre of the concentric equipotential contours moves horizontally away from the source. The axes of anisotropy remain unaltered.

CONCLUSION

An analytical solution for studying the effects of arbitrarily (oblique) anisotropic medium on electrical potential measurements as a result of a direct current distribution in the medium has been presented. If a mineral deposit is buried in or under an anisotropic layer, the success of locating the deposit (target) below the maximum (or minimum) on a profile on the plane of measurements (usually the earth's surface) would depend on the vertical depth between the target and the surface of measurements. This conclusion is based on the fact that electric point charges are developed on the surface of a target because of the conductivity contrast of the target and the host medium; depending on the depth of the charge distribution (the target), the centre of the elliptical confocal contours would move horizontally away from the target.

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