The Two-dimensional Green's Function for Electromagnetic Scattering

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Abstract

The usual integral form for the two-dimensional Green's function (a cosine transform) is evaluated in closed form. This alternative form is particularly useful when calculations have to be performed on a mini-computer.

1. Introduction

Because the number of mining companies owning mini-computers has increased, there is a corresponding increase in interest in the modelling of electromagnetic fields. A particular scheme for modelling the scattering of electromagnetic fields has been described by Hohmann (1971). The basis of the scheme is the solution of an integral equation, and to that end a Green's function must be evaluated many times. A particular feature of the Green's function is that it is the cosine transform of a function which depends on the vertical distance of a buried line source. In fact, the Green's function is also used to describe the electromagnetic fields about a line source. Because of the oscillating nature of the cosine function, the numerical integrations are slow and often inaccurate. The inaccuracies arise because it is usual to integrate between successive zeros of the cosine function and then to add all the terms together. The point is that when such a calculation is performed on a mini-computer, errors can arise because of numerical 'round-off'. The purpose of this paper is to present closed form expressions for the electric and magnetic Green's functions. By such means it is possible to avoid the problems mentioned above.

2. Electric Green's Function

The function used by Hohmann is $G(x^1,z^1;x,z)$ and it is defined by eqn 1:

$$G = \frac{-i\omega\mu_0}{2\pi}\int_0^\infty \left[\frac{(n_1-n_0)e^{-n_1(z+z^1)}}{(n_1+n_0)} + e^{-n_1|z-z^1|}\right]\frac{\cos(\lambda(x-x^1))d\lambda}{n_1} \qquad (1)$$

Here an $e^{i\omega t}$ time dependence has been assumed; the point (x^1,z^1) represents the source point while the point (x,z) denotes the position at which the function is to be evaluated. See Fig. 1.

$$n_1 = \sqrt{(\lambda^2+k_1^2)}, \qquad n_0 = \sqrt{(\lambda^2+k_0^2)}$$

$$k_1^2 = i\omega\mu_0\sigma_1 - \omega^2\mu_0\epsilon_1$$

$$k_0^2 = -\omega^2\mu_0\epsilon_0$$

The conductivity and permittivity of the ground are σ_1 and ϵ_1 respectively. The permittivity of the air is ϵ_0 and all permeabilities have been assumed to be the same as for free space, μ_0.

When displacement currents are neglected, eqn (1) may be reduced to:

$$G = \frac{-i\omega\mu_0\{K_0(k_1 R)+I\}}{2\pi} \qquad (2)$$

$$\text{where } I = \int_0^\infty \left(\frac{n_1-\lambda}{n_1+\lambda}\right)\frac{e^{-n_1(z+z^1)}}{n_1}\cos(\lambda(x-x^1))d\lambda$$

$$\text{and} \quad R = \sqrt{((z-z^1)^2 + (x-x^1)^2)}$$

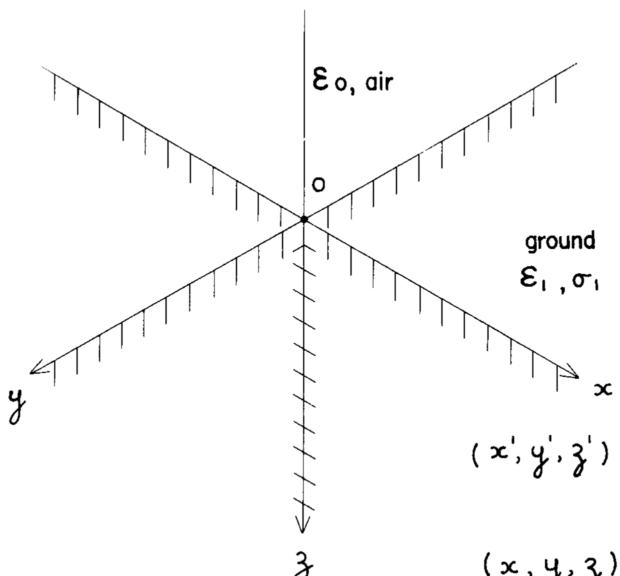

FIGURE 1
Geometry for the Green's Function

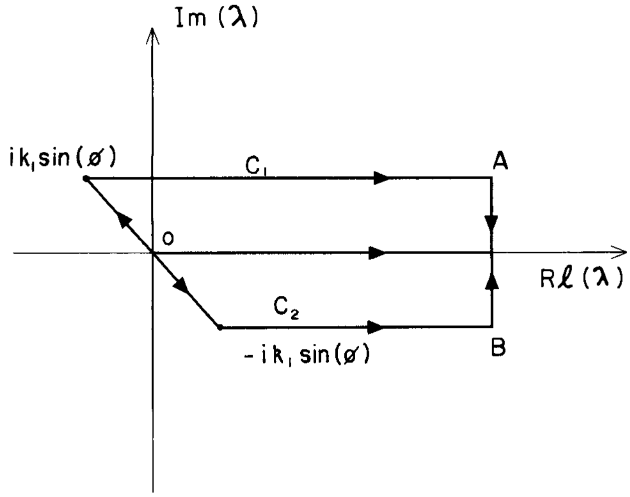

FIGURE 2
Path for the contour integrals

$K_0(z)$ is the modified Bessel function of argument z.

The remainder of this section is concerned with the evaluation of the term denoted by I.

To proceed, one notices that the term I is the sum of two terms denoted by I^+ and I^- where

$$I^+ = \frac{1}{2}\int_0^\infty \left(\frac{n_1-\lambda}{n_1+\lambda}\right)\frac{e^{-n_1(z+z^1)+i\lambda(x-x^1)}}{n_1}\,d\lambda$$

and

$$I^- = \frac{1}{2}\int_0^\infty \left(\frac{n_1-\lambda}{n_1+\lambda}\right)\frac{e^{-n_1(z+z^1)-i\lambda(x-x^1)}}{n_1}\,d\lambda \qquad (3)$$

In making this choice it is assumed that $(x-x^1) \geqslant 0$. In the event that this is not the case, the two terms are interchanged. Consequently it is possible to assume that the term $(x-x^1)$ is positive.

Notice that I^+ and I^- are analytic in the upper and lower λ planes with branch points at $\lambda = +ik_1$ and $-ik_1$ respectively. Also, the stationary points of

$$(z+z^1)n_1 \pm \lambda(x-x^1)i$$

are at

$$\lambda = \pm ik_1 \sin(\phi)$$

Here

$$(z+z^1) = P\cos(\phi);$$
$$(x-x^1) = P\sin(\phi)$$

and

$$P = \sqrt{((x-x^1)^2 + (z+z^1)^2)}$$
$$\phi = \arctan(|x-x^1|/(z+z^1))$$

These observations suggest that the paths of integration for I^+ or I^- be deformed to the paths C_1 and C_2 respectively.

To proceed, one divides the path of each integral into two parts. The first part is to $\pm ik_1 \sin(\phi)$ respectively, and the second part is to infinity via the points A and B respectively. The variable of integration is now changed in all of these integrals. In the integrals between 0 and $\pm ik_1\sin(\phi)$ one sets $\lambda = \pm ik_1 \sin(\alpha)$ respectively, and in the integrals between $\pm ik_1\sin(\phi)$ and infinity one writes $\lambda = \pm k_1 \sinh(\theta)$ respectively. Next, notice that the integral along AB vanishes because of Jordan's theorem.

By this means one finds that

$$I^+ + I^- = I = \int_0^\phi e^{-Pk_1\cos(\alpha-\phi)}\sin(2\alpha)d\alpha + \int_0^\infty e^{-Pk_1\cosh(\theta)-2\theta}\cosh(2\phi)d\theta \qquad (4)$$

The expression for I can now be reduced by straightforward integration to:

$$I = \cos(2\phi)\left[K_2(Pk_1) - \frac{2e^{-Pk_1\cos(\phi)}(1+Pk_1\cos(\phi))}{(Pk_1)^2}\right]$$
$$+\sin(2\phi)\int_0^\phi e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha \qquad (5)$$

Here we have made use of the integral representation for the modified Bessel function, i.e.

$$K_2(z) = \int_0^\infty e^{-z\cosh(\theta)}\cosh(2\theta)d\theta \qquad (6)$$

See Abramowitz & Stegun (1964, p. 376, No. 9-6-24).

The remaining integral may be expressed in terms of modified incomplete Struve functions of orders zero and one. The other quantities needed are modified and incomplete Bessel functions also of zero and unit order. A full discussion of this matter can be found in the book by Agrest and Maksimov (1971, p. 289). Since numerical results are required, however, it is easier to use eqn 5.

In terms of the initial rectangular co-ordinates one finds that the expression for the electric Green's function is:

$$G = \frac{-i\omega\mu_0}{2\pi}\left\{K_0(k_1R) + \right.$$
$$\frac{(z+z^1)^2-(x-x^1)^2}{P^2}\left[K_2(Pk_1) - \frac{2e^{-k_1(z+z^1)}(1+(z+z_1)k_1)}{(Pk_1)^2}\right]$$
$$\left. + \frac{2|x-x^1|(z+z^1)}{P^2}\left[\int_0^{\arctan\left(\frac{|x-x^1|}{(z+z^1)}\right)} e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha\right]\right\} \qquad (7)$$

3. The Magnetic Green's Function

Once the electric field is known it is a simple matter to calculate the associated magnetic fields. To this end we require the derivative of G with respect to x or z. In fact, if GH_z and GH_x are respectively the vertical and horizontal Green's function for the magnetic fields, then

$$GH_x = \frac{1}{i\omega\mu_0}\cdot\frac{\partial G}{\partial z}$$

and

$$GH_z = \frac{-1}{i\omega\mu_0}\cdot\frac{\partial G}{\partial x} \qquad (8)$$

After some tedious but not unpleasant work one finds that:

$$GH_x = -\frac{1}{2\pi}\Big\{ -\frac{(z-z^1)k_1}{R} K_1(k,R) +$$

$$e^{-k_1(z+z^1)} \Bigg[\frac{((z+z^1)^2-(x-x^1)^2)}{P^2} \Big\{ \frac{4k_1^2(z+z^1)}{(Pk_1)^4}(1+k_1(z+z^1))$$

$$+\frac{2k_1^2(z+z^1)}{(Pk_1)^2}\Big\} - \frac{8(z+z^1)(x-x^1)^2}{P^4}\left(\frac{1+k_1(z+z^1)}{(Pk_1)^2}\right)\Bigg]$$

$$+ K_0(Pk_1)\Big\{\frac{3(z+z^1)}{P^4}\left(2(x-x^1)^2-(z+z^1)^2\right)\Big\}$$

$$+ K_1(k_1P)\Big\{\frac{8(z+z^1)(x-x^1)^2}{k_1 R^5} - \frac{((z-z^1)^2-(x-x^1)^2)\,k_1(z+z^1)}{P^3}\left(1+\frac{4}{(Pk_1)^2}\right)\Big\}$$

$$+\left(\frac{2|x-x^1|}{P^2} - \frac{4|x-x^1|(z+z^1)^2}{R^4}\right)\int_0^{\arctan\left(\frac{|x-x^1|}{(z+z^1)}\right)} e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha$$

$$-2\frac{|x-x^1|(z+z^1)^2}{P^3}\int_0^{\arctan\left(\frac{|x-x^1|}{(z+z^1)}\right)} k_1 e^{-k_1 P\cos(\alpha)}\cos(\alpha)\cos(2\alpha)d\alpha$$

$$-\frac{2(x-x^1)^2(z+z^1)}{P^6}\left((z+z^1)^2-(x-x^1)^2\right)e^{-k_1(z+z^1)}\Big\} \qquad (9)$$

and $(x \gtrless x_1)$

$$GH_z = \pm\frac{1}{2\pi}\Big\{\frac{\mp(x-x^1)k_1}{R} K_1(k_1R) +$$

$$e^{-k_1(z+z^1)}\left(1+k_1(z+z^1)\right)\Bigg[\frac{4(x-x^1)k_1^2}{(Pk_1)^4P^2}\left((z+z^1)^2-(x-x^1)^2\right)$$

$$+\frac{8(x-x^1)(z+z_1)^2}{P^4(Pk_1)^2}\Bigg] \frac{+2e^{-k_1(z+z^1)}((x-x^1)\left((z+z^1)^2-(x-x^1)^2\right)(z+z^1)^2}{P^6}$$

$$-K_1(Pk_1)\Bigg[\frac{8(x-x^1)(z+z^1)^2}{P^5k_1} (+ \frac{\left((z+z^1)^2-(x-x^1)^2\right)k_1(x-x^1)}{P^3} \cdot$$

$$\cdot\left(1+\frac{4}{(Pk_1)^2}\right)\Bigg]$$

$$-K_0(Pk_1)\left[\frac{2(x-x^1)\left((z+z^1)^2-(x-x^1)^2\right)}{P^4} + \frac{4(x-x^1)(z+z^1)^2}{P^4}\right]$$

$$+\left(\frac{2(z+z^1)}{P^2} - \frac{4(x-x^1)^2(z+z^1)}{P^4}\right)\int_0^{\arctan\left(\frac{(x-x^1)}{(z+z^1)}\right)} e^{-Pk_1\cos(\alpha)}\cos(2\alpha)d\alpha$$

$$-\frac{2(x-x^1)^2(z+z^1)k_1}{P^3}\int_0^{\arctan\left(\frac{(x-x^1)}{(z+z^1)}\right)} e^{-Pk_1\cos\alpha}\cos(\alpha)\cos(2\alpha)d\alpha\Big\} \qquad (10)$$

4. Discussion

The expressions given here have uses other than the evaluation of the Green's function. Thus, the electromagnetic fields about line sources on or in a uniform ground can be expressed in terms of the Green's functions that have just been described. More specifically, if a current of $I_0e^{i\omega t}$ flows in a wire, then the electric field E is just

$$E = I_0e^{i\omega t}G \qquad (11)$$

Associated with the electric field are two components of a magnetic field H_x and H_z. These quantities are simply

$$H_x = I_0e^{i\omega t}GH_x$$

and

$$H_z = I_0e^{i\omega t}GH_z \qquad (12)$$

For a discussion of results alternative to those given here, the reader is referred to Wait & Spies (1971). There the interested reader will find a table of G for various values of x and z.

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