Paper

The Effect of Displacement Currents on Time Domain Electromagnetic Fields

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Abstract

The usual practice of neglecting displacement currents in the calculation of electromagnetic transients is critically examined for the case of a uniform ground. For this case it is shown that the percentage error involved in the calculation by the neglect of the displacement currents is approximately (1500/4) arctan $[\epsilon_1/(\sigma_1 t)]$, where t denotes time and σ_1 and ϵ_1 denote the conductivity and permittivity of the ground respectively.

1. Introduction

Recently there has been a growing interest in the possibility of detecting induced polarization effects in transient electromagnetic method (TEM) data. In a review of the literature Lee (1979) noted that there were few theoretical results that might provide an insight into this problem. More recently Spies (1980) published results of a TEM survey in Queensland. Spies concluded that induced polarization effects might account for the negative responses which were observed in some of the transient data of that survey.

Later Lee (1981) showed that induced polarization effects were present in the transient response of a ground whose conductivity function could be described by a 'Cole-Cole' model (Pelton et al. 1978). In all these results the effects of displacement currents were ignored. The purpose of the present paper is to show that that procedure is valid except for some extreme cases.

2. The Transient Response

Suppose that a large loop of radius a lies on a uniform ground of conductivity σ_1 and permittivity ϵ_1. When a uniform current of spectrum $-I_0 e^{-i\omega t}/(i\omega)$ flows in the loop an electric field $\bar{E}(\omega)$ will be observed.

Morrison, Phillips & O'Brien (1969 p. 87, eqn 22) have shown that at a distance r from the centre of the loop, $\bar{E}(\omega)$ can be written as:

$$\bar{E}(\omega) = \frac{I_0 a\mu_0}{2\pi}\int_0^{2\pi}\int_0^{\infty} J_0(\lambda R)\,\frac{\lambda}{n_0}\,\frac{n_0\, e^{-i\omega t}}{n_0 + n_1}\cos(\theta)\, d\lambda\, d\theta \quad (1)$$

Notice that we have used the addition theorem of the Bessel functions to group the functions into a single term.

In the above equation: $R = \sqrt{r^2 + a^2 - 2ar\cos(\theta)}$
$n_j = \sqrt{\lambda^2 - k_j^2}$
$k_0^2 = \omega^2\mu_0\epsilon_0$
and $k_1^2 = \omega^2\mu_0\epsilon_1 + i\omega\mu_0\sigma_1$

An assumption made is that the permeabilities of the air and ground are the same as the permeability of free space, μ_0. The notation ϵ_0 is for the permittivity of the air; ω denotes angular frequency, and t time.

Equation (1) may be conveniently written as:

$$\bar{E}(\omega) = \frac{-a\mu_0 I_0 e^{-i\omega t}}{(k_0-k_1)(k_0+k_1)}\int_0^{2\pi}\int_0^{\infty}\frac{(\lambda n_0 - \lambda n_1)J_0(\lambda R)\,d\lambda\,\cos(\theta)\,d\theta}{2\pi} \quad (2)$$

With the help of Sommerfield's integral one can reduce eqn (2) to eqn (3):

$$\bar{E}(\omega) = \frac{-a\mu_0 I_0 e^{-i\omega t}}{(k_0-k_1)(k_0+k_1)}\int_0^{2\pi}\frac{\partial^2}{\partial z^2}\left(\frac{e^{ik_0P}}{P} - \frac{e^{ik_1P}}{P}\right)\frac{\cos(\theta)\,d\theta}{2\pi} \quad (3)$$

Here $P = \sqrt{R^2 + z^2}$ and z is set equal to zero once the differentiation is carried out.

The transient electric field E(t) may be found by taking the Fourier transform:

$$E(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\bar{E}(\omega)\, d\omega$$

This transform may be evaluated by contour integration once a suitable path is selected in the lower complex ω plane.

Notice that there is no pole at $k_0 = k_1$, because the exponential terms are also zero there. Also, there is not a pole at $k_0 = -k_1$. Thus it remains only to consider the branch points of $\bar{E}(\omega)$ in the lower complex ω plane. In the following we shall assume that E(t) is to be evaluated for times greater than $\sqrt{\epsilon_1\mu_0}P$ and $\sqrt{\epsilon_0\mu_0}P$. This assumption is because we are interested only in those times for which the electric field has had time to propagate to all the points on the loop.

Since $k_1 = \sqrt{\omega^2\mu_0\epsilon_1 + i\omega\mu_0\sigma_1}$ one sees that there are branch points at $\omega = 0$ and $\omega = -i\sigma_1/\epsilon_1$. The spectrum of

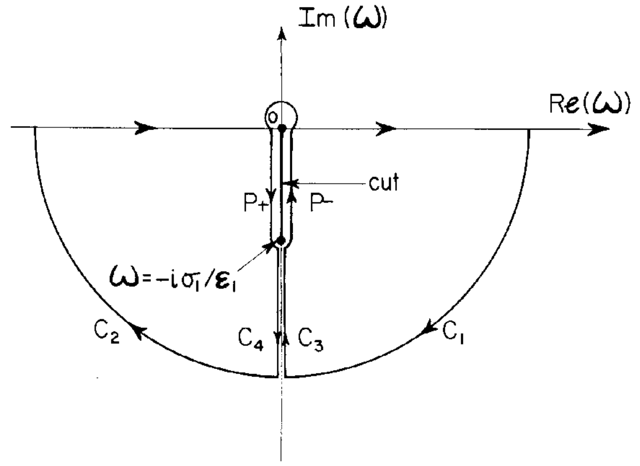

FIGURE 1
Path for the contour integral.

$\bar{E}(\omega)$ is an analytic function of ω provided we make a cut along the ray from $\omega = 0$ to $\omega = -i\sigma_1/\epsilon_1$. We have ignored the point at infinity because it is a removable singularity (see Sveshnikov & Tikhonov 1973, p. 18).

Notice that the spectrum of $\bar{E}(\omega)$ vanishes exponentially for ω tending to infinity because of the exponential functions. Moreover, the integrals of $\bar{E}(\omega)$ along the arcs C_1 and C_2 of Fig. 1 vanish because of Jordan's lemma (see Sveshnikov & Tikhonov 1973, p. 130). Notice, also, that the integrals along the paths C_3 and C_4 of Fig. 1 cancel each other as the paths of integration approach the negative ω axis. Also, the integral about the origin of that figure vanishes as the path of integration approaches the origin because the spectrum of $\bar{E}(\omega)$ is finite there. Thus by Cauchy's theorem the only contribution to the contour integral is the integral about the branch cut from $\omega = 0$ to $\omega = -i\sigma_1/\epsilon_1$.

To proceed one writes: $\omega = qe^{-i\pi/2}$ on P–
and $\omega = qe^{i\pi + i\pi/2}$ on P+
and learns that $k_1 = \pm\sqrt{q\mu_0\sigma_1 - q^2\epsilon_1\mu_0}$ on P– and P+ respectively.

Notice that the integrals about the ends of the contours vanish and so after adding the integrals along each side of the cut together, one has from Cauchy's theorem that:

$$E(t) = \frac{aI_0\mu_0}{2\pi^2}\int_0^{2\pi}\frac{\partial^2}{\partial z^2}\frac{\cos(\theta)}{P}\int_0^{\sigma_1/\epsilon_1}\frac{e^{-qt}\sin(P\sqrt{q\mu_0\sigma_1 - q^2\mu_0\epsilon_1})}{q^2\mu_0(\epsilon_1-\epsilon_0) - q\mu_0\sigma_1}\,dq\,d\theta \tag{4}$$

The integration with respect to θ may be performed by first interchanging the order of integration, expanding the sine function as a power series and then differentiating with respect to z at z = 0. The resulting θ integral may be evaluated by using the double-angle formulae for the trigonometric functions. These manipulations lead to the expression for E(t) given in eqn (5):

$$E(t) = \frac{aI_0\mu_0}{\mu_0\pi}\int_0^{\sigma_1/\epsilon_1}\frac{e^{-qt}}{q^2(\epsilon_1-\epsilon_0)-q\sigma_1}\sum_{s=0}^{\infty}\frac{(-1)^s a^{2s+2}(q\mu_0\sigma_1 - q^2\mu_0\epsilon_1)^{\frac{2s+5}{2}}}{(2s+5)\,(2s+3)s!\,(s+2)!}\,dq \tag{5}$$

A convenient check can be placed on this result by setting $\epsilon_1 = \epsilon_0 = 0$. For this case $E(t) = E_P(t)$ where:

$$E_P(t) = -\frac{aI_0}{\sigma_1\pi}\int_0^{\infty}\frac{e^{-qt}}{q}\sum_{s=0}^{\infty}\frac{(-1)^s a^{2s+2}(q\mu_0\sigma_1)^{\frac{2s+5}{2}}}{(2s+5)\,(2s+3)s!\,(s+2)!}\,dq \tag{6}$$

This last expression can be integrated term by term by using the definition of the gamma function. This integration yields:

$$E_P(t) = -\frac{aI_0\mu_0\sqrt{\pi}}{2\pi t}\sqrt{\frac{\sigma_1\mu_0}{t}}\sum_{s=0}^{\infty}\frac{(-1)^s(2s+2)!}{(2s+5)\,(s+1)!s!\,(s+2)!}\left(\frac{a^2\sigma_1\mu_0}{4t}\right)^{s+1} \tag{7}$$

Equation (7) is identical to that given by Lee & Lewis (1974, eqn 11).

3. Effects of Displacement Currents

Equation (5) may be transformed into a more convenient form by the following substitutions:

$$\begin{aligned} qt &= \tan\theta \\ \phi &= \arctan[\epsilon_1/(\sigma_1 t)] \\ \alpha &= (\epsilon_1 - \epsilon_0)/\epsilon_1 \end{aligned} \tag{8}$$

One now finds that eqn (5) can be written as:

$$E(t) = \frac{I_0}{\sigma_1\pi a^2}\int_0^{\frac{\pi}{2}-\phi}\frac{e^{-\tan\theta}\sec^2\theta}{\alpha\tan\phi - \cot\theta}\cdot$$

$$\sum_{s=0}^{\infty}(-1)^s\left(\frac{a^2\mu_0\sigma_1}{t}\right)^{\frac{2s+5}{2}}\frac{\tan^{2s+3}\theta}{(2s+5)\,(2s+3)s!\,(s+2)!}\left(\frac{\cos(\theta+\phi)}{\cos\phi\,\sin\theta}\right)^{\frac{2s+5}{2}}d\theta \tag{9}$$

Notice that when $\phi = 0$:

$$E(t) = E_P(t) \tag{10}$$

For the problem we are considering we may suppose that $\epsilon_1 \gg \epsilon_0$, in which case α can be taken to be unity.

Since we are interested only in the effects of displacement currents we may restrict our attention to the case where σ_1 is small. For that case, for all the times used in prospecting, eqn (7) may be approximated by the first term.

Consequently: $E(t) \approx \left(1 - \dfrac{\phi z^2\partial^2}{\partial z^2}\right) E_P(t)$ where $z = 1/t$ (11)

and therefore: $E(t) \approx [1 - (15/4)\phi]\,E_P(t)$ (12)

Since the voltage induced in a receiving loop which is coincident with the transmitting loop is just $2\pi a\,E(t)$, it follows that:

$$V(t) \approx [1 - (15/4)\phi]\;V_P(t) \tag{13}$$

Here $V_P(t)$ denotes the transient voltage when displacement currents are ignored. Thus if ϕ is small the percentage error χ that would result in the transient electric field (or measured voltage) due to the neglect of the displacement currents is approximated by:

$$\chi = (1500/4)\arctan[\epsilon_1/(\sigma_1 t)] \tag{14}$$

Since $\epsilon_0 = 8.854 \times 10^{-12}$ F.m^{-1}, one finds that for the case where ϵ_1 is 100 times ϵ_0 and σ, a and t have values of 10^{-3} S.m^{-1}, 50 m and 10^{-4} s respectively, the resultant percentage error is about 4%.

4. Discussion

Equation (14) shows that only at the very early stages of the transient response of a highly resistive rock would the effect of displacement currents be seen.

Instruments with the capability of measuring transients at times less than 0.1 ms have a potential application for mapping rocks with a large $\epsilon_1/(\sigma_1 t)$: for example, coal seams.

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